



Effect of Variable Prestress on Natural Frequencies of Rayleigh Beams under Travelling Distributed Loads

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ABSTRACT

This research is concerned with analysing the effect of variable prestress on natural frequencies of finite Rayleigh beams under distributed loads. Analytical solutions which often highlight the vibrating structure is obtained for the simply supported boundary condition. The solution technique is based on the Generalized Galerkin method and modification of Strubles' asymptotic method. The focus of the study is particularly on the effect of some structural parameters such as variable prestressed function and foundation constant on the natural frequencies. From the results, it is evident that the increase in Prestress values gives rise to a corresponding increase in both natural and modified natural frequencies. Thus, it can be deduced that resonance is reached earlier for lower values of prestress as well as lower values of foundation constant. For greater values of foundation constant (within a good soil-structure interaction), the likelihood for collapse of the Rayleigh beam structure is quite remote.

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INTRODUCTION

Historically, design methods for unbonded post-tensioned concrete flexural members have assumed that the effective pre-stress force is constant for the full length of the tendon [Rudloff (2006), Saiididi et al (1994), Law and Lu (2005), Oni (2005)]. Designers theorized that the variable tendon force that exists due to friction at the time of stressing distributes with time until the force becomes equal at all points in the tension. Attempts on vibrational problems of prestressed structures subjected to distributed loads were made by Kim and Chung (2009), Adetunde (2007), Omolofe and Ogunyebi (2009) and Andi (2017). In the above investigations, the pre-stress is assumed to be constant. However, a study published by Burns et al (1991) concludes that the assumed re-distribution of tendon forces

does not occur and that the design of unbonded post-tensioned members should be based on a variable effective pre-stress force. Generally speaking, variable prestress is of much significance and mainly arises from several physical backgrounds including gravity load in the gravity field, centrifugal force when dealing with a rotating beam, etc. There are major mathematical complications inherent in a variable force approach: calculating the functions of the prestress force becomes much more complex if the force is variable (Bondy 1992). Researchers that carried out studies on variable pre-stress include Omolofe and Ajibola (2005) who investigated the effect of variable axial force on the dynamic response of Bernoulli-Euler beam subjected to concentrated moving load, with analytical solutions provided only for the simply supported boundary conditions.

Kocatürk and Simsek (2006) employed Lagrange's equations to investigate the dynamic response of eccentrically prestressed viscoelastic Timoshenko beams under a moving harmonic load. The trial functions denoting deflection of the beam and the rotation of the cross-sections were expressed in polynomial forms. The effects of the value of the eccentricity of the compressive load, the excitation frequency, constant velocity of the transverse moving harmonic load and viscous damping of the material of beams were studied in detail. Much later, Li et al (2013) considered the bending vibration of a cantilever with an attached concentrated mass at the free end. Arbitrarily axial compressive or tensile loading was applied. Also the influence of rotatory inertia of cross-section was taken into account in the analysis. The associated boundary value problem was converted to an integral equation to obtain the natural frequencies. The effects of axial loading, tip mass and rotatory inertia on the natural frequencies were discussed. Ogunyebi (2014) investigated the effect of variable prestress on the behavior of thin beams on elastic foundation. The load was moving distributed. It was seen that the higher the values of the axial force N , and foundation modulus K , the lower the response amplitude of the beam. Sun and Li (2019) studied the free vibration of non-uniformly axially functionally graded Timoshenko beams subjected to combine axially tensile or compressive loading. The initial value method was developed to determine the natural frequencies. Influence of shear

rigidity, taper ratio, gradient index, tip force and axially distributed loading on the natural frequencies of the clamped-free beams were discussed. It was found that material inhomogeneity and geometric non-uniform cross section strongly affect higher order vibration frequencies and mode shapes.

This work therefore concerns the effect of variable prestress on natural frequencies of finite Rayleigh beams under distributed loads. The solution technique is based on the Generalized Galerkin method, the use of the expression of Heaviside function in series form and a modification of the Struble's asymptotic method. Analytical solutions for both the moving force and moving mass models for the simply supported boundary condition is obtained. Basically, more often than not, prestressed concrete members are precast in nature but not all precast concrete sections are prestressed; hence the usual simply supported boundary condition for prestressed concrete members. Also practically, precast conditions are neither adequate nor apt enough for continuous beams. Hence, the research centres on the simply supported boundary condition. Illustrative examples in which the effects of variable prestressed function and foundation constant on the natural frequencies are herein considered as well as the conditions under which resonance occurs in the vibrating system. Figure 1 shows a typical appearance of a prestress beam.



Figure 1- A Typical appearance of a Prestress Beam



MATHEMATICAL MODEL

The dynamic response to a moving distributed load of a finite Rayleigh beam is

$$EI \frac{\partial^4 V(x, t)}{\partial x^4} + \mu \frac{\partial^2 V(x, t)}{\partial t^2} - \mu R^0 \frac{\partial^4 V(x, t)}{\partial x^2 \partial t^2} - \frac{\partial}{\partial x} \left[N(x) \frac{\partial V(x, t)}{\partial x} \right] + KV(x, t) = MgH(x - vt) \left[1 - \frac{1}{g} \left(\frac{\partial^2 V(x, t)}{\partial t^2} + 2v \frac{\partial^2 V(x, t)}{\partial x \partial t} + v^2 \frac{\partial^2 V(x, t)}{\partial x^2} \right) \right] \quad (1)$$

where x is the spatial coordinate, $V(x, t)$ is the transverse displacement, E is the Young's modulus, I is the moment of inertia of the beam, R^0 is the measure of rotatory inertia effect, μ is the constant mass per unit length of the beam. K is the

governed by the fourth order partial differential equation (Fryba 1972),

elastic foundation constant, $N(x)$ is the variable axial force, $H(x - vt)$ is the Heaviside function and g is the acceleration due to gravity. t is assumed to be limited to that interval of time within which the mass μ is on the beam, that is

$$0 \leq vt \leq L \quad (2)$$

v is velocity of the distributed mass and L is length of the beam,

The end supports for the beam are simply supported and the associated initial conditions are

$$V(x, 0) = V_t(x, 0) = 0 \quad (3)$$

As an example, let the variable axial force be defined as

$$N(x) = N_0 \left(1 + \sin \frac{\pi x}{L} \right)^3 \quad (4)$$

where N_0 is the average value of the axial force of the elastic beam.

Substituting equation (4) into equation (1) simplifying and rearranging, one obtains

$$EI \frac{\partial^4 V(x, t)}{\partial x^4} + \mu \frac{\partial^2 V(x, t)}{\partial t^2} - \mu R^0 \frac{\partial^4 V(x, t)}{\partial x^2 \partial t^2} - \frac{\partial}{\partial x} \left[\frac{N_0}{4} \left(10 + 15 \sin \frac{\pi x}{L} - 6 \cos \frac{2\pi x}{L} - \sin \frac{3\pi x}{L} \right) \frac{\partial V(x, t)}{\partial x} \right] + KV(x, t) + H(x - vt) \left(\frac{\partial^2 V(x, t)}{\partial t^2} + 2v \frac{\partial^2 V(x, t)}{\partial x \partial t} + v^2 \frac{\partial^2 V(x, t)}{\partial x^2} \right) = MgH(x - vt) \quad (5)$$

Equation (5) is the fourth order non-homogeneous partial differential equation with variable coefficients.

SOLUTION PROCEDURE

In this section, the procedure developed to solve equation (5) involves expressing the Heaviside function as a Fourier series and employing an elegant approximation technique due to Galerkin referred to as Generalized Galerkin Method

(GGM) to simplify and reduce equation (5) to a sequence of second order ordinary differential equations. These second order equations are then treated using a modification of the Struble's asymptotic method.

Generalized Galerkin Method (GGM)

The technique requires that the solution of equation (5) be of the form,

$$V_n(x, t) = \sum_{m=1}^n W_m(t) U_m(x) \quad (6)$$

where $U_m(x)$ is chosen such that the desired boundary conditions are satisfied. The function $U_m(x)$ is chosen as the beam function,

$$U_m(x) = \sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} + B_m \sinh \frac{\lambda_m x}{L} + C_m \cosh \frac{\lambda_m x}{L} \quad (7)$$

so that

$$U_k(x) = \sin \frac{\lambda_k x}{L} + A_k \cos \frac{\lambda_k x}{L} + B_k \sinh \frac{\lambda_k x}{L} + C_k \cosh \frac{\lambda_k x}{L} \quad (8)$$

The constants $A_m, B_m, C_m, A_k, B_k, C_k$ and the mode frequencies λ_m, λ_k are obtained using the appropriate boundary conditions.

Simply supported boundary conditions

The Rayleigh beam with simple supports at both ends has the following boundary conditions,

$$V_n(0, t) = 0 = V_n(L, t), \quad \frac{\partial^2 V_n(0, t)}{\partial x^2} = 0 = \frac{\partial^2 V_n(L, t)}{\partial x^2} \quad (9)$$

and hence for normal modes,

$$U_m(0) = 0 = U_m(L), \quad \frac{\partial^2 U_m(0)}{\partial x^2} = 0 = \frac{\partial^2 U_m(L)}{\partial x^2} \quad (10)$$

which implies that,

$$U_k(0) = 0 = U_k(L), \quad \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2} \quad (11)$$

Applying (10) and (11),

$$A_m = A_k = 0, \quad B_m = B_k = 0, \quad C_m = C_k = 0 \quad (12)$$

$$\lambda_m = m\pi, \quad \lambda_k = k\pi \quad (13)$$

Simplification

Equation (6) when substituted into equation (5), yields

$$\sum_{m=1}^n \left\{ EIU_m''(x)W_m(t) + \mu U_m(x)\dot{W}_m(t) - \mu R^0 U_m''(x)\dot{W}_m(t) - \frac{N_0}{4} \left(10 + 15 \sin \frac{\pi x}{L} - 6 \cos \frac{2\pi x}{L} - \sin \frac{3\pi x}{L} \right) U_m''(x)W_m(t) + KU_m(x)W_m(t) + MH(x-vt) \left(U_m(x)\dot{W}_m(t) + 2vU_m'(x)\dot{W}_m(t) + v^2 U_m''(x)W_m(t) \right) \right\} - MgH(x-vt) = 0 \quad (14)$$

In order to determine $W_m(t)$, it is required that the expression on the left hand side of equation (9) be orthogonal to the function $U_k(x)$. Hence,

$$\int_0^L \left\{ \sum_{m=1}^n \left\{ EIU_m''(x)W_m(t) + \mu U_m(x)\dot{W}_m(t) - \mu R^0 U_m''(x)\dot{W}_m(t) - \frac{N_0}{4} \left(10 + 15 \sin \frac{\pi x}{L} - 6 \cos \frac{2\pi x}{L} - \sin \frac{3\pi x}{L} \right) U_m''(x)W_m(t) + KU_m(x)W_m(t) + MH(x-vt) \left(U_m(x)\dot{W}_m(t) + 2vU_m'(x)\dot{W}_m(t) + v^2 U_m''(x)W_m(t) \right) \right\} - MgH(x-vt) \right\} U_k(x) dx = 0 \quad (15)$$

Rearranging equation (15) one obtains,

$$\sum_{m=1}^n \left\{ (T_0 - T_1) \ddot{W}_m(t) + T_2 W_m(t) + \frac{M}{\mu} [T_3(t) \dot{W}_m(t) + 2vT_4(t) \dot{W}_m(t) + v^2 T_5(t) W_m(t)] \right\} = \frac{Mg}{\mu} T_6(t) \quad (16)$$

where,

$$T_0 = \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (17)$$

$$T_1 = -R^0 \left(\frac{m\pi}{L} \right)^2 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (18)$$

$$T_2 = T_{2A} + T_{2B} - T_{2C} - T_{2D} + T_{2E} + T_{2F} \quad (19)$$

$$T_{2A} = \frac{K}{\mu} \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (20)$$

$$T_{2B} = -\frac{10N_0}{4\mu} \left(\frac{m\pi}{L} \right)^2 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (21)$$

$$T_{2C} = -\frac{15N_0}{4\mu} \left(\frac{m\pi}{L} \right)^2 \int_0^L \sin \frac{\pi x}{L} \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (22)$$

$$T_{2D} = -\frac{6N_0}{4\mu} \left(\frac{m\pi}{L} \right)^2 \int_0^L \cos \frac{2\pi x}{L} \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (23)$$

$$T_{2E} = -\frac{N_0}{4\mu} \left(\frac{m\pi}{L} \right)^2 \int_0^L \sin \frac{3\pi x}{L} \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (24)$$

$$T_{2F} = -\frac{EI}{\mu} \left(\frac{m\pi}{L} \right)^4 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (25)$$

$$T_3(t) = \int_0^L H(x - vt) U_m(x) U_k(x) dx \quad (26)$$

$$T_4(t) = \int_0^L H(x - vt) U'_m(x) U_k(x) dx \quad (27)$$

$$T_5(t) = \int_0^L H(x - vt) U''_m(x) U_k(x) dx \quad (28)$$

$$T_6(t) = \int_0^L H(x - vt) U_k(x) dx \quad (29)$$

The ordinary differential equation (16) is valid for the simply supported boundary condition.

In order to evaluate the integrals in (26) to (29), use is made of the Fourier series representation for the Heaviside unit step function namely,

$$H(x - vt) = \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi(x-vt))}{2n+1}, \quad 0 < x < L \quad (30)$$

Substituting (7), (8) and (30) into (16) and simplifying yields,

$$\sum_{m=1}^n \left\{ T_{10}(m, k) \ddot{W}_m(t) + T_2(m, k) W_m(t) + \frac{M}{\mu} \left[Q_{1A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{1B}(n, m, k) \right] \right\}$$

$$\begin{aligned}
 & -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{1c}(n, m, k) \Big) \dot{W}_m(t) \\
 & \quad + 2v \left(Q_{2A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{2B}(n, m, k) \right. \\
 & -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{2c}(n, m, k) \Big) \dot{W}_m(t) \\
 & \quad + v^2 \left(Q_{3A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{3B}(n, m, k) \right. \\
 & \left. -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{3c}(n, m, k) \right) W_m(t) \Big] = \frac{MgL}{\mu k \pi} \left[-\cos k\pi + \cos \frac{k\pi v}{L} t \right] \quad (31)
 \end{aligned}$$

where,

$$T_{10}(m, k) = T_0 - T_1 \quad (32)$$

$$Q_{1A}(m, k) = \frac{1}{4} \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (33)$$

$$Q_{1B}(n, m, k) = \int_0^L \sin(2n+1)\pi x \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (34)$$

$$Q_{1c}(n, m, k) = \int_0^L \cos(2n+1)\pi x \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (35)$$

$$Q_{2A}(m, k) = \frac{m\pi}{4L} \int_0^L \cos \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (36)$$

$$Q_{2B}(n, m, k) = \frac{m\pi}{L} \int_0^L \sin(2n+1)\pi x \cos \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (37)$$

$$Q_{2c}(n, m, k) = \frac{m\pi}{L} \int_0^L \cos(2n+1)\pi x \cos \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (38)$$

$$Q_{3A}(m, k) = -\frac{(m\pi)^2}{4L^2} \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (39)$$

$$Q_{3B}(n, m, k) = -\left(\frac{m\pi}{L}\right)^2 \int_0^L \sin(2n+1)\pi x \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (40)$$

$$Q_{3c}(n, m, k) = -\left(\frac{m\pi}{L}\right)^2 \int_0^L \cos(2n+1)\pi x \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (41)$$

Rearranging equation (31), one obtains

$$\begin{aligned}
 & \sum_{m=1}^n \{T_{10}(m, k) \dot{W}_m(t) + T_2(m, k) W_m(t) \\
 & \quad + \eta_0 L \left[\left(Q_{1A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{1B}(n, m, k) \right. \right. \\
 & -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{1c}(n, m, k) \Big) \dot{W}_m(t) \\
 & \quad \left. \left. + 2v \left(Q_{2A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{2B}(n, m, k) \right. \right. \right.
 \end{aligned}$$



$$\begin{aligned}
 & -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{2c}(n, m, k) \Bigg) \dot{W}_m(t) \\
 & + v^2 \left(Q_{3A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{3B}(n, m, k) \right. \\
 & \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{3c}(n, m, k) \right) W_m(t) \Bigg] = \frac{MgL}{\mu\lambda_k} \left[-\cos k\pi + \cos \frac{k\pi v}{L} t \right] \quad (42)
 \end{aligned}$$

where,

$$\eta_0 = \frac{M}{\mu L} \quad (43)$$

Equation (42) is now the fundamental equation governing the problem of the dynamic behaviour of finite Rayleigh beams with variable prestress under moving distributed loads. This non-homogeneous second order ordinary differential equation holds for the simply supported boundary condition. In what follows, two special cases of equation (42) are considered.

SOLUTION OF THE FUNDAMENTAL EQUATION

Case I: *Simply supported Rayleigh beam with variable prestress subjected to moving distributed force*

The differential equation describing the dynamic behaviour of a finite simply supported Rayleigh beam with variable prestress subjected to a moving distributed force is obtained from equation (42) by setting $\eta_0 = 0$. In this case, we obtain

$$\sum_{m=1}^n \{T_{10}(m, k) \ddot{W}_m(t) + T_2(m, k) W_m(t)\} = \frac{MgL}{\mu k \pi} \left[-\cos k\pi + \cos \frac{k\pi v}{L} t \right] \quad (44)$$

Rearranging

equation (44) and ignoring the summation sign yields,

$$\ddot{W}_m(t) + \beta_{mf}^2 W_m(t) = P_{mf} \left[-\cos k\pi + \cos \frac{k\pi v}{L} t \right] \quad (45)$$

where

$$\beta_{mf}^2 = \frac{T_2(m, k)}{T_{10}(m, k)} \quad (46a)$$

is the natural frequency corresponding to frequency of the free system due to the presence of the moving force.

Evaluating the integrals in (46a) we have,

$$\begin{aligned}
 \beta_{mf}^2 = & \frac{K}{\mu} \left(\frac{L^2}{L^2 + R^0 m^2 \pi^2} \right) - \frac{20N_0 m^2 \pi^2}{16L\mu} \left(\frac{2L}{L^2 + R^0 m^2 \pi^2} \right) + \frac{15N_0 m^2 \pi}{16L\mu} \left(\frac{2L}{L^2 + R^0 m^2 \pi^2} \right) \\
 & \cdot \left(\frac{\cos(1+2m)\pi - 1}{1+2m} + \frac{\cos(1-2m)\pi - 1}{1-2m} + 4 \right) - \frac{N_0 m^2 \pi^2}{8\mu L^2} \left(\frac{2L}{L^2 + R^0 m^2 \pi^2} \right) \\
 & \cdot \left(\frac{2L}{3\pi} + \frac{L}{2\pi} \left(\frac{\cos(3+2m)\pi - 1}{3+2m} + \frac{\cos(3-2m)\pi - 1}{3-2m} \right) \right) + \frac{EI m^4 \pi^4}{2\mu L^3} \left(\frac{2L}{L^2 + R^0 m^2 \pi^2} \right) \quad (46b)
 \end{aligned}$$

And,



$$P_{mf} = \frac{MgL}{\mu k \pi T_{10}(m, k)} \quad (47a)$$

or,

$$P_{mf} = \frac{2Mg}{\mu k \pi \left[1 + \frac{R^0 m^2 \pi^2}{L^2} \right]} \quad (47b)$$

Solving equation (45) in conjunction with the initial conditions, one obtains expression for $W_m(t)$. Thus in view of (6),

$$V_n(x, t) = \frac{2}{L} \sum_{m=1}^n \left\{ P_{mf} \left[\frac{\text{Cos} \alpha_v t - \text{Cos} \beta_{mf} t}{\beta_{mf}^2 - \alpha_v^2} - \text{Cos} k \pi \left(\frac{1 - \text{Cos} \beta_{mf} t}{\beta_{mf}} \right) \right] \cdot \text{Sin} \frac{m \pi x}{L} \right\} \quad (48)$$

where,

$$\alpha_v = \frac{k \pi v}{L} \quad (49)$$

We can rewrite (48) as,

$$V_n(x, t) = \frac{2}{L} \sum_{m=1}^n \left\{ P_{mf} \left[\frac{\text{Cos} \alpha_v t - \text{Cos} \beta_{mf} t}{\beta_{mf}^2 - \alpha_v^2} + \frac{(-1)^k \text{Cos} \beta_{mf} t}{\beta_{mf}} - \frac{(-1)^k}{\beta_{mf}} \right] \cdot \text{Sin} \frac{m \pi x}{L} \right\} \quad (50)$$

Equation (50) represents the transverse displacement response to a moving distributed force of the simply supported finite Rayleigh beam with variable prestress.

Case II: Simply supported Rayleigh beam with variable prestress subjected to moving distributed mass

If the inertia effect of the moving mass is not negligible, then, $\eta_0 \neq 0$ and the solution to the entire equation (42) is required. This gives the moving mass

$$\begin{aligned} \dot{W}_m(t) + \frac{2v\eta_0 LR_2(n, m, k)}{T_{10}(m, k) + \eta_0 LR_1(n, m, k)} \dot{W}_m(t) + \frac{T_2(m, k) + v^2 \eta_0 LR_3(n, m, k)}{T_{10}(m, k) + \eta_0 LR_1(n, m, k)} W_m(t) \\ = \frac{\eta_0 L^2 g}{k \pi} \left(-\text{Cos} k \pi + \text{Cos} \frac{\lambda_k v t}{L} \right) \end{aligned} \quad (51)$$

where,

$$\begin{aligned} R_1(n, m, k) = Q_{1A}(m, k) \\ + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\text{Cos}(2n+1)\pi v t}{2n+1} Q_{1B}(n, m, k) \\ - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\text{Sin}(2n+1)\pi v t}{2n+1} Q_{1C}(n, m, k) \end{aligned} \quad (52)$$

problem. It is observed that an analytical solution to this equation is not readily available. In order to get analytical solution, use is made of the modified asymptotic method due to Struble discussed in (Nayfeh 1973). Equation (42) is thus simplified and rearranged to take the form

$$\begin{aligned}
 R_2(n, m, k) &= Q_{2A}(m, k) \\
 &+ \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{2B}(n, m, k) \\
 &- \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{2C}(n, m, k) \quad (53)
 \end{aligned}$$

$$\begin{aligned}
 R_3(n, m, k) &= Q_{3A}(m, k) \\
 &+ \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{3B}(n, m, k) \\
 &- \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{3C}(n, m, k) \quad (54)
 \end{aligned}$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (51). Thus, the right hand side of (51) is set to zero, and a parameter $\eta_1 < 1$ is considered for any arbitrary mass ratio η_0 defined by,

$$\eta_1 = \frac{\eta_0}{1 + \eta_0} \quad (55)$$

It can be shown that,

$$\eta_0 = \eta_1 + o(\eta_1^2) \quad (56)$$

and

$$\frac{1}{T_{10}(m, k) + \eta_1 LR_1(n, m, k)} = \frac{1}{T_{10}(m, k)} \left[1 - \frac{\eta_1 LR_1(n, m, k)}{T_{10}(m, k)} \right] + o(\eta_1^2) \quad (57)$$

where,

$$\left| \frac{\eta_1 LR_1(n, m, k)}{T_{10}(m, k)} \right| < 1 \quad (58)$$

when η_1 is set to zero in equation (51) a situation corresponding to the case when the inertia effect of the mass of the system is neglected is obtained. Then the solution of (51) can be written as,

$$W_m(t) = \psi_m \cos(\beta_m t - \theta_m) \quad (59)$$

where β_m is as defined previously and ψ_m and θ_m are constants.

Furthermore as $\eta_1 < 1$, Struble's technique requires that the asymptotic solution to the homogeneous part of equation (51) be of the form,

$$W_m(t) = \psi(m, t) \cos(\beta_m t - \theta(m, t)) + \eta_1 W_1(m, t) + o(\eta_1^2) \quad (60)$$

where $\psi(m, t)$ and $\theta(m, t)$ are slowly varying functions of time.

Substituting equation (60) and its derivatives into homogeneous part of equation (51) and taking into account (57) while neglecting the terms in $o(\eta_1^2)$, one obtains

$$\begin{aligned}
 &-2\beta_m \dot{\psi}(m, t) \sin(\beta_m t - \theta(m, t)) + 2\beta_m \psi(m, t) \dot{\theta}(m, t) \cos(\beta_m t - \theta(m, t)) \\
 &- \psi(m, t) \beta_m^2 \cos(\beta_m t - \theta(m, t)) \\
 &- \frac{2v\eta_1 L \psi(m, t)}{T_{10}(m, k)} \left[Q_{2A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} Q_{2B}(n, m, k) \right. \\
 &\quad \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{2C}(n, m, k) \right] \beta_m.
 \end{aligned}$$



$$\begin{aligned} & \sin(\beta_m t - \theta(m, t)) + \frac{T_2(m, k)}{T_{10}(m, k)} \psi(m, t) \cos(\beta_m t - \theta(m, t)) \\ & + \frac{v^2 \eta_1 L}{T_{10}(m, k)} \left[Q_{3A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi vt}{2n+1} \right. \\ & \left. Q_{3B}(n, m, k) - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi vt}{2n+1} Q_{3C}(n, m, k) \right] \psi(m, t) \cos(\beta_m t - \theta(m, t)) = 0 \quad (61) \end{aligned}$$

The variational equations are obtained by equating the coefficients of $\cos(\beta_m t - \theta(m, t))$ and $\sin(\beta_m t - \theta(m, t))$ terms on

both sides of equation (61) to zero. Hence, noting the following trigonometric identities,

$$\begin{aligned} & \frac{\cos(2n+1)\pi vt}{2n+1} \sin[\beta_m t - \theta(m, t)] \\ & = \frac{1}{2} \sin \left[\frac{(2n+1)\pi vt}{2n+1} + \beta_m t - \theta(m, t) \right] - \frac{1}{2} \sin \left[\frac{(2n+1)\pi vt}{2n+1} - \beta_m t + \theta(m, t) \right] \\ & \frac{\sin(2n+1)\pi vt}{2n+1} \sin[\beta_m t - \theta(m, t)] \\ & = \frac{1}{2} \cos \left[\frac{(2n+1)\pi vt}{2n+1} - \beta_m t + \theta(m, t) \right] - \frac{1}{2} \cos \left[\frac{(2n+1)\pi vt}{2n+1} + \beta_m t - \theta(m, t) \right] \\ & \frac{\cos(2n+1)\pi vt}{2n+1} \cos[\beta_m t - \theta(m, t)] \\ & = \frac{1}{2} \cos \left[\frac{(2n+1)\pi vt}{2n+1} + \beta_m t - \theta(m, t) \right] + \frac{1}{2} \cos \left[\frac{(2n+1)\pi vt}{2n+1} - \beta_m t + \theta(m, t) \right] \\ & \frac{\sin(2n+1)\pi vt}{2n+1} \cos[\beta_m t - \theta(m, t)] = \frac{1}{2} \sin \left[\frac{(2n+1)\pi vt}{2n+1} + \beta_m t - \theta(m, t) \right] + \frac{1}{2} \sin \left[\frac{(2n+1)\pi vt}{2n+1} - \beta_m t + \theta(m, t) \right] \quad (62) \end{aligned}$$

and neglecting the terms that do not contribute to the variational equations, equation (61) reduces to,

$$\begin{aligned} & -2\beta_m \dot{\psi}(m, t) \sin(\beta_m t - \theta(m, t)) + 2\beta_m \psi(m, t) \dot{\theta}(m, t) \cos(\beta_m t - \theta(m, t)) - \frac{2v\eta_1 L \psi(m, t)}{T_{10}(m, k)} \\ & Q_{2A}(m, k) \beta_m \sin(\beta_m t - \theta(m, t)) + \frac{v^2 \eta_1 L}{T_{10}(m, k)} Q_{3A}(m, k) \psi(m, t) \cos(\beta_m t - \theta(m, t)) \\ & = 0 \quad (63) \end{aligned}$$

The variational equations are respectively,

$$2\beta_m \psi(m, t) \dot{\theta}(m, t) + \frac{v^2 \eta_1 L}{T_{10}(m, k)} Q_{3A}(m, k) \psi(m, t) = 0 \quad (64)$$

and,

$$2\beta_m \dot{\psi}(m, t) + \frac{2v\eta_1 L \psi(m, t)}{T_{10}(m, k)} Q_{2A}(m, k) \beta_m = 0 \quad (65)$$

Solving equations (64) and (65) respectively, one obtains,

$$\theta(m, t) = -\frac{v^2 \eta_1 L Q_{3A}(m, k)}{2\beta_m T_{10}(m, k)} t + \theta_m \quad (66)$$

where θ_m is a constant and $\psi(m, t) = A_0 e^{-\chi_0 t}$, where A_0 is a constant and,

$$\chi_0 = \frac{v\eta_1 L Q_{2A}(m, k)}{T_{10}(m, k)} \quad (67)$$

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogeneous system is,

$$W_m(t) = \psi(m, t) \cos(\Phi_{mm}t - \theta_m) \quad (68)$$

where,

$$\Phi_{mm} = \frac{2\beta_m^2 T_{10}(m, k) + v^2 \eta_1 L Q_{3A}(m, k)}{2\beta_m T_{10}(m, k)} \quad (69)$$

or,

$$\Phi_{mm} = \frac{8\beta_{mf}^2 (L^2 + R^0 m^2 \pi^2) - v^2 \eta_1 L m^2 \pi^2}{8\beta_{mf} (L^2 + R^0 m^2 \pi^2)} \quad (69a)$$

is called the modified natural frequency corresponding to the frequency of the free system due to the presence of the moving mass. Thus, to solve the non-homogeneous equation (51), the differential operator which acts on $W_m(t)$ is replaced by the equivalent free system operator defined by the modified frequency Φ_{mm} ,

$$\ddot{W}_m(t) + \Phi_{mm}^2 W_m(t) = P_{mm} \left(-\cos k\pi + \cos \frac{k\pi v}{L} t \right) \quad (70)$$

where,

$$\Phi_{mm} = \frac{8\beta_{mf}^2 (L^2 + R^0 m^2 \pi^2) - v^2 \eta_1 L m^2 \pi^2}{8\beta_{mf} (L^2 + R^0 m^2 \pi^2)} \quad (71)$$

Equation (70) is analogous to equation (45). Thus, using similar argument as in Case I, $W_m(t)$ can be obtained and which on inversion yields,

$$V_n(x, t) = \frac{2}{L} \sum_{m=1}^n P_{mm} \left[\frac{\cos \alpha_v t - \cos \Phi_{mm} t}{\Phi_{mm}^2 - \alpha_v^2} + \frac{(-1)^k \cos \Phi_{mm} t}{\Phi_{mm}} - \frac{(-1)^k}{\Phi_{mm}} \right] \sin \frac{m\pi x}{L} \quad (72)$$

This represents the transverse displacement response to a moving distributed mass of the simply supported finite Rayleigh beam with variable prestress.

DISCUSSION OF ANALYTICAL SOLUTIONS

If the undamped system such as this is studied, it is desirable to examine the response amplitude of the dynamical system which may grow without bound since beam failure can occur either by concrete crushing or by reinforcement

$$\beta_{mf} = \frac{k\pi v}{L} \quad (73)$$

while equation (72) indicates that the same prestressed beam under the action of a moving distributed mass experiences resonance effect when

$$\Phi_{mm} = \frac{k\pi v}{L} \quad (74)$$

Evidently, from equation (69a),

$$\Phi_{mm} = \beta_{mf} \left[1 - \frac{\eta_1}{8\beta_{mf}^2} \left(\frac{v^2 m^2 \pi^2}{L^2 + R^0 m^2 \pi^2} \right) \right] = \frac{k\pi v}{L} \quad (75)$$

From equations (73) and (75), it is evident that for the same natural frequency, the critical speed for the system consisting of a prestressed simply

yielding. This we call resonance conditions. Equation (50) clearly shows that the prestressed simply supported finite Rayleigh beam traversed by a moving distributed force reaches a state of resonance whenever

supported finite Rayleigh beam traversed by a moving distributed force at constant speed is greater than that of the moving distributed mass problem. Thus, resonance

is reached earlier in the moving distributed force system.
 mass system than in the moving distributed

NUMERICAL CALCULATION AND DISCUSSION

Numerical results for the analysis presented above are depicted in plotted curves where the following are beam parameters,

S/N	PARAMETERS
1.	Density (ρ)=2400kg/m ³
2.	Modulus of Elasticity (E) = 0.17 x 10 ¹¹ N/mm ²
3.	Span (L) = 10m
4.	Width (b) = 2m
5.	Depth (d) = 0.5m
6.	Mass per unit length (μ) = 2400kg/m
7.	Cross sectional area (A) = 1m ²
8.	Weight of the beam = 24 kN/m
9.	Velocity of travelling distributed body (v) = 2m/s
10.	Moment of Inertia (I) = 0.0417m ⁴
11.	Rotary Inertia correction factor (R^0) = 0.0417m ²
12.	Foundation constant (K) = 400,000
13.	Modified Mass ratio $\eta_1 = 0.5$
14.	Iteration number (m) = 1, 2, 3

Discussion of plotted curves

Figure 2 depicts the graph of Natural frequency against corresponding Prestress for different foundation constants K= 0, 4000, and 40,000. From the graph, it

is observed that the Natural frequency increases with Prestress. Thus, for higher values of Prestress, our beam design assumes more stability.

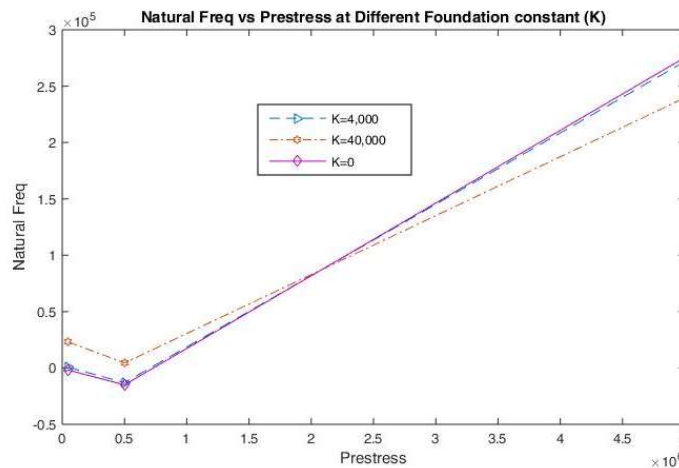


Figure 2: Graph of Natural Frequency against Prestress for different Foundation constants K

In Figure 3, it is seen that the Modified natural frequency is proportional to the prestress indicating more stability

and reliability in the beam with increased prestress values. From Figure 2 and Figure 3, it is evident that the increase in Prestress

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values gives rise to a corresponding natural frequencies.
 increase in both natural and modified

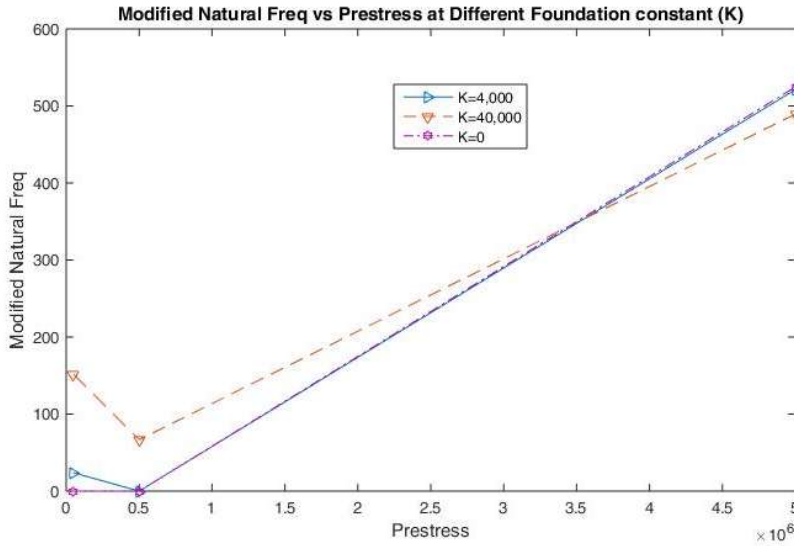


Figure 3: Graph of Modified Natural Frequency against Prestress for different Foundation constants K

Figure 4 shows the plotted curves of Natural frequency against foundation constants all at a prestress value of $N_0 = 50,000$. The graph shows that the

relationship between the natural frequency and foundation constant is strongly proportionate.

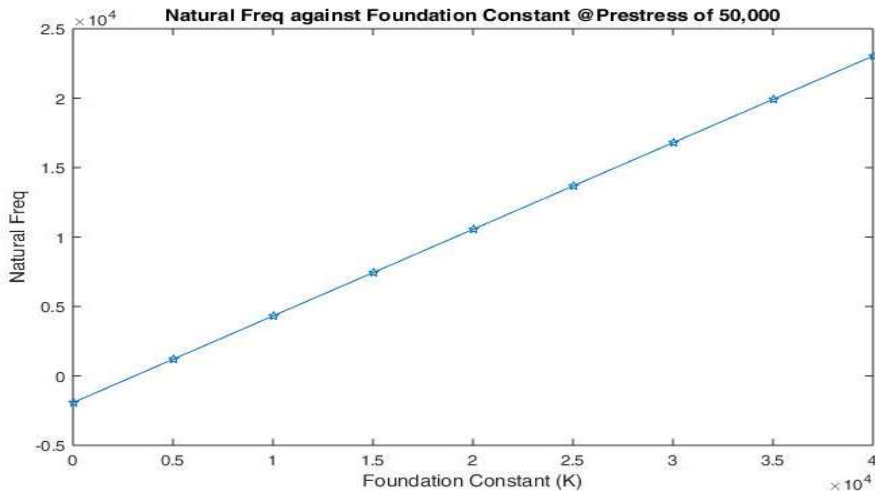


Figure 4: Graph of Natural Frequency against Foundation constant K at $N_0 = 50,000$

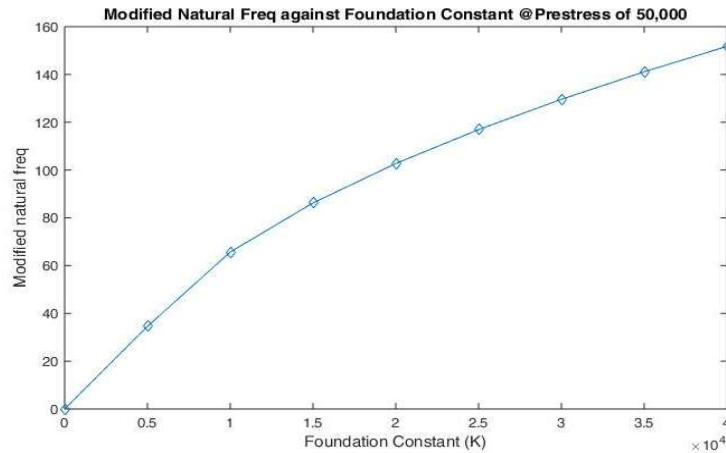


Figure 5: Graph of Modified Natural Frequency against Foundation constant K at $N_0 = 50,000$

Figure 5 which is a graph of Modified Natural frequency against foundational constant shows a similar pattern to Figure 4 all at a common prestress value of $N_0 = 50,000$. This is reflective of the fact that resonance is reached earlier for lower values of foundation constants (stiffness).

CONCLUSION

This research analyzed the effect of variable prestress on natural frequencies of finite Rayleigh beams under distributed loads. Analytical solutions for the problem were obtained for the simply supported boundary condition. The solution technique was based on the Generalized Galerkin method and modification of Strubles' asymptotic method. The focus of the study was particularly on the effect of some structural parameters such as variable prestressed function and foundation constant on the natural frequencies. The results showed that increase in Prestress values gave rise to a corresponding increase in both natural and modified natural frequencies. Thus, it can be deduced that resonance is reached earlier for lower values of prestress as well as lower values

of foundation constants K. It can be ultimately hypothesised that the Rayleigh beam design is more stable with a remote risk of resonance for high values of prestress. Also, for greater values of foundation constant (within a good soil-structure interaction), the likelihood for collapse of the Rayleigh beam structure is quite remote. It is evident that the beam structure is likely to be more safe and stable considering attaining resonance as well as vulnerability to collapse at its natural frequency than at its modified natural frequency from all the plots.

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