



Best Proximity Points of Cyclic Contraction Mappings in M - Normal Cone Metric Spaces

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ABSTRACT

In this paper, using the notion of Banach, Kannan and Charterjea contraction mappings, we introduce new contraction mapping and prove the existence and convergence of best proximity point in the context of complete M - Normal Cone Metric spaces.

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INTRODUCTION

The study of nonlinear operators had its beginning about the start of the twentieth century with the investigation into the existing properties of solutions to certain boundary value problems arising in ordinary and partial differential equations. The earliest techniques, largely devised by [1], involved the iteration of an integral operator to obtain solutions to such problems. In 1922, these techniques were given precise abstract formulation by [2] and [3] contraction mappings principle, a powerful tool in analysis for establishing existence and uniqueness of the solution of problems in a different kind. The fact that a fixed point theorem is an important tool for solving the equation of the form $Tx = x$ where T is a self-map defined on a subset of some suitable space, leads to the significance of this area. And is useful in the theory of Newtonian and non-Newtonian calculus. Various authors have generalized the Banach-Cacciopoli contraction principle

in different spaces. [4] characterized metric completeness by generalized Banach Contraction principle. [5] first introduced the notion of cone metric space, [6] introduced the notion of cone rectangular metric space, proved Banach contraction mapping principles and some fixed point theorems of such mappings on a complete cone metric space. [7] studied common fixed points for weak commutative mappings on a cone metric space. Moreover, [22] Introduced the concept of M - normal cone and proved fixed point results for several contraction mappings. [21] prove Common fixed Points for a class of Contraction Mapping in a Metric space. [8] studied some unique fixed point theorems in the set up of cone metric space. Their results carry fixed point results of [9] and [10] to cone metric spaces. For more details about fixed point see [[15], [16], [17], [18], [19], [20] e.t.c]. However, if T is not self-mapping, then it is probable that the equation $Tx = x$ has no solution. In this case, [11] introduced the concept of the

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best approximation theorems which is a concept that explores the existence of an approximate solution, that is, if A is a nonempty subset of considered space X and $T: A \rightarrow X$ then we can find a point $x \in A$ such that $d(x, Tx) = d(Tx, A)$. On the other hand, the best proximity point theorem was also considered by different authors in the context of metric space, which is a concept that analyzed the

existence of an approximate solution that is optimal; that is, if A and B are non-empty subsets of considered space and $T: A \cup B \rightarrow A \cup B$ then we can find a point $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf_{n \in \mathbb{N}} \{d(x, y) : x \in A \text{ and } y \in B\}$. Let A and B be nonempty closed subsets of a complete multiplicative metric space X . A mapping $T: A \cup B \rightarrow A \cup B$ satisfying $T(A) \subseteq B, T(B) \subseteq A$ and

$$d(Tx, Ty) \leq d(x, y)^\lambda \cdot d^*(A, B)^{1-\lambda}, \quad (1)$$

for all $x \in A$ and $y \in B$, is called multiplicative cyclic contraction mappings. The results were motivated by the observation that if for some $\lambda \in (0, 1)$, Equation (1) holds, then $A \cap B = \emptyset$ and so T has a unique fixed point in X . [12] introduced a generalization of Equation (1) to extend this case to when $A \cap B \neq \emptyset$, which does not entail $A \cap B$ to be non-empty and asked for the best proximity point instead of fixed-point; i.e a point $x \in A$ such that $d^*(x, Tx) = d^*(A, B)$. [12], first proved the following interesting best proximity point theorem concerning cyclic contraction maps in metric spaces. Also, [12], proved the above best proximity point theorem in the set up of multiplicative metric space.

Theorem 1.1 [12] *Let A and B be non-empty closed subsets of a multiplicative metric space X and let $T: A \cup B \rightarrow A \cup B$ be a multiplicative cyclic contraction map. Fix $x_1 \in A$ and define $x_{n+1} = Tx_n, n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A . Then $\exists x \in A$ such that $d^*(x, Tx) = d^*(A, B)$.*

Let us recall that a function $\phi: [0, \infty) \rightarrow [0, 1]$ is said to be an MT function if $\limsup_{s \rightarrow t^+} \phi(s) < 1$ for all $t \in [0, \infty)$. By using MT-function, [13] introduced the concept of MT-cyclic contraction map and proved the following theorem.

Theorem 1.2 [13] *Let A and B be non-empty subsets of a metric space (X, d) and*

$T: A \cup B \rightarrow A \cup B$ be an MT-cyclic contraction concerning ϕ . Let $x_1 \in A$ be given and define an iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Tx_n, n \in \mathbb{N}$. Suppose that $\{x_{2n-1}\}$ has a convergent subsequence in A . Then there exists $v \in A$ such that $d^(v, Tv) = d^*(A, B)$.*

In [12], the above result was also proved in multiplicative metric space. Under some suitable conditions, proved the following best proximity point theorem.

Theorem 1.3 [12] *Let A and B be non-empty subsets of a multiplicative metric space (X, d^*) . Let $T: A \cup B \rightarrow A \cup B$ be a multiplicative MT cyclic contraction concerning ϕ . Let $x_1 \in A$ be given and define an iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Tx_n, n \in \mathbb{N}$. Suppose that $\{x_{2n-1}\}$ has a convergent subsequence in A . Then there exists $v \in A$ such that $d^*(v, Tv) = d^*(A, B)$.*

The best proximity point theorem for a subclass of generalised MT-cyclic contraction mappings was established in [14] in the context of metric space. In [14], the following subclass of cyclic mappings was introduced.

Definition 1.4 *Let A and B be non-empty subsets of a metric space (X, d) . A cyclic map $T: A \cup B \rightarrow A \cup B$ is called generalised cyclic MT-contraction mapping if there exists an MT-function ϕ such that*

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$$d(Tx, Ty) \leq \varphi(d(x, y)) \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)] + \min\{d(x, Ty), d(y, Tx)\}\} + (1 - \varphi(d(x, y)))d(A, B)$$

for all $x \in A$ and $y \in B$. [14], also established the following theorem that is related to generalised cyclic MT - contraction mapping.

Theorem 1.5 Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \rightarrow A \cup B$ be a generalised MT -cyclic contraction. For each $x_1 \in A$ define an iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Tx_n$. Then $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = d(A, B)$.

It is obvious that if $\phi : [0, \infty) \rightarrow [0, 1]$ is a non-decreasing function or a non-increasing function, then ϕ is an MT -function. So the set of MT function is a rich class.

Thus, motivated by the results above, one of the main objectives of this thesis is to consider a wider class of such introduced mappings which was considered by [12]. Also, some new existence theorems of such mappings will be studied.

Proposition 1.6 [14] Let $\phi : [0, \infty) \rightarrow [0, 1]$ be a function. Then, ϕ is an MT -function if and only if for any non-increasing sequence $\{x_n\}_{n \in (0, \infty)}$,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_n) + (1 - \alpha)d(A, B) \\ &\leq \alpha d(x_{n-1}, x_n) - \alpha d(x_{n-1}, x_n) + d(x_{n-1}, x_n) \\ &= d(x_{n-1}, x_n) \end{aligned}$$

Hence, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence in $(0, 1)$ Now, Since T is a cyclic contractions, we have

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq \alpha d(x_1, x_2) + (1 - \alpha)d(A, B) \\ &= \alpha d(x_1, x_2) + d(A, B) - \alpha d(A, B) \\ &\leq \alpha d(x_1, x_2) + d(A, B) \end{aligned}$$

(2)

and

$$\begin{aligned} d(x_3, x_4) &= d(Tx_2, Tx_3) \\ &\leq \alpha d(x_2, x_3) + (1 - \alpha)d(A, B) \\ &\leq \alpha[\alpha d(x_1, x_2) + d(A, B)] + (1 - \alpha)d(A, B) \end{aligned}$$

MAIN RESULTS

In this section, best proximity point of cyclic contraction mapping is established.

Theorem 2.1 Let A and B be two non-empty subsets of a complete Cone Metric Space X and P an M - normal Cone. Suppose the mapping $T : A \cup B \rightarrow A \cup B$ is a cyclic contraction. For each $x_1 \in A$ define an iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Tx_n$. Then

$$i. \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = d(A, B).$$

i.i Suppose that $\{x_{2n-1}\}_{n \in \mathbb{N}}$ has a convergent subsequence in A , then there exists $x \in A$ s.t $d(x, Tx) = d(A, B)$.

Proof.

Since T is a cyclic contraction, then there exists $\alpha \in (0, 1)$ s.t $d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B)$ for all $x \in A$ and $y \in B$. Now, choose any point $x_0 \in X$ s.t $\{x_n\}_{n \in \mathbb{N}}$ in X is defined by, $x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \dots, x_{n+1} = Tx_n, \dots$ s.t if $x_n \in A$ and $Tx_n \in B$ and if $x_{n+1} \in B, Tx_{n+1} \in A$. we shall show that $\{d(x_n, x_{n+1})\}$ is monotone non-increasing,

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$$\begin{aligned}
 &= \alpha^2 d(x_1, x_2) + \alpha d(A, B) + d(A, B) - \alpha d(A, B) \\
 &= \alpha^2 d(x_1, x_2) + d(A, B)
 \end{aligned}$$

Also, (3)

$$\begin{aligned}
 d(x_4, x_5) &= d(Tx_3, Tx_4) \\
 &\leq \alpha d(x_3, x_4) + (1 - \alpha)d(A, B) \\
 &\leq \alpha[\alpha^2 d(x_1, x_2) + d(A, B)] + (1 - \alpha)d(A, B) \\
 &= \alpha^3 d(x_1, x_2) + \alpha d(A, B) + d(A, B) - \alpha d(A, B) \\
 &= \alpha^3 d(x_1, x_2) + d(A, B)
 \end{aligned}$$

(4)

Inductively, we obtain $d(A, B) \leq d(x_n, x_{n+1}) \leq \alpha^n d(x_1, x_2) + d(A, B)$ for all $n \in \mathbb{N}$. Since $\alpha \in [0, 1)$, $\lim_{n \rightarrow \infty} \alpha^n = 0$. Using (3) above and nonincreasing nature of $\{d(x_n, x_{n+1})\}$, we obtain $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \text{dist}(A, B)$. ii. Let $\{x_{2nk-1}\}$ be a subsequence of $\{x_{2n-1}\}$ s.t $x_{2nk} \rightarrow x$ as $k \rightarrow \infty$. Now, observe that $d(A, B) \leq d(x, x_{2nk}) \leq d(x, x_{2nk-1}) + d(x_{2nk-1}, x_{2nk})$ for all $n \in \mathbb{N}$. using $\lim_{k \rightarrow \infty} d(x, x_{2nk-1}) = 0$ and the inequality (4) above, we take limit as $k \rightarrow \infty$ in inequality (5),

to have $d(A, B) \leq \lim_{k \rightarrow \infty} d(x, x_{2nk}) \leq \lim_{k \rightarrow \infty} d(x, x_{2nk-1}) + \lim_{k \rightarrow \infty} d(x_{2nk-1}, x_{2nk})$. This implies, $d(A, B) \leq \lim_{k \rightarrow \infty} d(x, x_{2nk}) \leq d(A, B)$.

Which also implies, $\lim_{k \rightarrow \infty} d(x, x_{2nk}) = d(A, B)$. Now,

$$\begin{aligned}
 d(A, B) &\leq d(x_{2nk+1}, Tx) \\
 &= d(x_{2nk}, Tx) \\
 &\leq \alpha d(x_{2nk}, x) + (1 - \alpha)d(A, B) \\
 &\leq \alpha d(x_{2nk}, x) + (1 - \alpha)d(x_{2nk}, x) \\
 &= d(x_{2nk}, x)
 \end{aligned}$$

and this implies, $d(A, B) \leq d(x_{2nk+1}, Tx) \leq d(x_{2nk}, x)$.

Taking limit as $k \rightarrow \infty$, we have

$$d(A, B) \leq d(x, Tx) \leq d(A, B)$$

and so

$$d(x, Tx) = d(A, B).$$

Theorem 2.2 Let A and B be two non-empty subsets of a complete Cone Metric Space X and P an M -normal Cone. For $c \in E$, with $0 << c$, $x_0 \in X$ and the set of open ball $Br(x_0) = \{x \in X : d(x_0, x) < c\}$. Suppose the mapping $T : A \cup B \rightarrow A \cup B$ satisfies the cyclic contraction condition $d(Tx, Ty) \leq \alpha^{\lambda} d(x, y) + (1 - \alpha^{\lambda})c$, for all $x, y \in Br(x_0)$ where $0 \leq \alpha < 1$,

$0 < \lambda < 1$ and $d(Tx_0, x_0) < (1 - \alpha^{\lambda})c$. Then

i. There exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$

$$\begin{aligned}
 d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
 &\leq \alpha^{\lambda} d(x_{n-1}, x_n) + (1 - \alpha^{\lambda})d(A, B) \\
 &\leq \alpha^{\lambda} d(x_{n-1}, x_n) - \alpha^{\lambda} d(x_{n-1}, x_n) + d(x_{n-1}, x_n)
 \end{aligned}$$

s.t $d(x_n, x_{n+1}) \rightarrow d(A, B)$.

i.i Suppose that the sequence $\{x_{n-1}\}_{n \in \mathbb{N}}$ has a convergent subsequence in A , then there exists $x \in A$ s.t $d(x, Tx) = d(A, B)$.

Proof.

i. For $A \cap B = \emptyset$ and $\alpha = 0$, the result holds trivially.

Suppose that $\alpha^{\lambda} \neq 0$ and $A \cap B = \emptyset$. Firstly, let show that the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing,

$$= d(x_{n-1}, x_n)$$

Thus, $\{d(x_n, x_{n+1})\}$ is non-increasing sequence Now,

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq \alpha^\lambda d(x_1, x_2) + (1 - \alpha^\lambda)d(A, B) \\ &= \alpha^\lambda d(x_1, x_2) + d(A, B) - \alpha^\lambda d(A, B) \\ &\leq \alpha^\lambda d(x_1, x_2) + d(A, B) \end{aligned}$$

And

$$\begin{aligned} d(x_3, x_4) &= d(Tx_2, Tx_3) \\ &\leq \alpha^\lambda d(x_2, x_3) + (1 - \alpha^\lambda)d(A, B) \\ &\leq \alpha^\lambda [\alpha^\lambda d(x_1, x_2) + d(A, B)] + (1 - \alpha^\lambda)d(A, B) \\ &= \alpha^{2\lambda} d(x_1, x_2) + \alpha^\lambda d(A, B) + d(A, B) - \alpha^\lambda d(A, B) \\ &= \alpha^{2\lambda} d(x_1, x_2) + d(A, B) \end{aligned} \tag{5}$$

Also,

$$\begin{aligned} d(x_4, x_5) &= d(Tx_3, Tx_4) \\ &\leq \alpha^\lambda d(x_3, x_4) + (1 - \alpha^\lambda)d(A, B) \\ &\leq \alpha^\lambda [\alpha^{2\lambda} d(x_1, x_2) + d(A, B)] + (1 - \alpha^\lambda)d(A, B) \\ &= \alpha^{3\lambda} d(x_1, x_2) + \alpha^\lambda d(A, B) + d(A, B) - \alpha^\lambda d(A, B) \\ &= \alpha^{3\lambda} d(x_1, x_2) + d(A, B) \end{aligned} \tag{6}$$

Continuing in this manner, we obtain

$$d(A, B) \leq d(x_n, x_{n+1}) \leq \alpha^{n\lambda} d(x_1, x_2) + d(A, B) \tag{7}$$

Taking limit as $\lim_{n \rightarrow \infty} \alpha^n = 0$, we get

$$d(A, B) \leq \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) \leq \lim_{n \rightarrow \infty} [\alpha^{n\lambda} d(x_1, x_2) + d(A, B)]$$

This implies,

$$d(A, B) \leq \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) \leq d(A, B) \tag{8}$$

and this implies,

$$d(x_n, x_{n+1}) \rightarrow \text{dist}(A, B).$$

ii. Let the subsequence of $\{x_n - 1\}$ be $\{x_{nk-1}\}$, then $x_{nk-1} \rightarrow x$ in A as $k \rightarrow \infty$.

Now, observe that

$$d(A, B) \leq d(x, x_{nk}) \leq d(x, x_{nk-1}) + d(x_{nk-1}, x_{nk})$$

for all $n \in \mathbb{N}$.

Therefore taking limit as $k \rightarrow \infty$, we have

$$d(A, B) \leq \lim_{k \rightarrow \infty} d(x, x_{nk}) \leq \lim_{k \rightarrow \infty} d(x, x_{nk-1}) + \lim_{k \rightarrow \infty} d(x_{nk-1}, x_{nk})$$

$$d(A, B) \leq \lim_{k \rightarrow \infty} d(x, x_{nk}) \leq 0 + \lim_{k \rightarrow \infty} d(x_{nk-1}, x_{nk})$$

$$d(A, B) \leq \lim_{k \rightarrow \infty} d(x, x_{nk}) \leq \lim_{k \rightarrow \infty} d(x_{nk-1}, x_{nk})$$

$$d(A, B) \leq \lim_{k \rightarrow \infty} d(x, x_{nk}) \leq d(A, B)$$

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Which also implies,

$$\lim_{k \rightarrow \infty} d(x, x_{nk}) = d(A, B)$$

Now,

$$\begin{aligned} d(A, B) &\leq d(x_{nk+1}, Tx) \\ &= d(x_{nk}, Tx) \\ &\leq \alpha^\lambda d(x_{nk}, x) + (1 - \alpha^\lambda) d(A, B) \\ &\leq \alpha^\lambda d(x_{nk}, x) + (1 - \alpha^\lambda) d(x_{nk}, x) \\ &= d(x_{nk}, x) \end{aligned} \tag{11}$$

and this implies,

$$d(A, B) \leq d(x_{nk+1}, Tx) \leq d(x_{nk}, x)$$

Taking limit as $k \rightarrow \infty$, we have

$$d(A, B) \leq \lim_{k \rightarrow \infty} d(x_{nk+1}, Tx) \leq \lim_{k \rightarrow \infty} d(x_{nk}, x)$$

$$d(A, B) \leq d(x, Tx) \leq d(A, B)$$

and so

$$d(x, Tx) = d(A, B).$$

Corollary 2.3 ([22]). *Let X be a complete Cone Metric Space, P an M normal Cone. Suppose the mapping $T : X \rightarrow X$ is a contraction. Then, T has a unique fixed point in X .*

Proof.

By setting $A \cup B = X$ in Theorem 2.1 the result follows.

Corollary 2.4 ([22]). *Let X be a complete Cone Metric Space, P an M - normal Cone. For $c \in E$, with $0 << c$, $x_0 \in X$ and the set of open ball*

$Br(x_0) = \{x \in X : d(x_0, x) < r\}$. Suppose the mapping $T : X \rightarrow X$ satisfies the cyclic contraction condition $d(Tx, Ty) \leq \alpha^\lambda d(x, y)$, for all $x, y \in Br(x_0)$ where $0 \leq \alpha < 1$, $0 < \lambda < 1$ and $d(Tx_0, x_0) < (1 - \alpha^\lambda)r$. Then, T has a unique fixed point in $Br(x_0)$.

Proof.

Let $A \cup B = X$ and the mapping $d(A, B) = 0$ in Theorem 2.2, then the result follows.

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