



Error Analysis of the Tau Method for a Class of third Order Initial Value Problem

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ABSTRACT

In this paper, the differential variant of the tau method and its error estimation for initial value problem of the special class of third order overdetermined ordinary differential equation is considered. The error estimation is based on the error polynomial of the economisation of power series. The n th degree approximant is derived for the estimated error using the initial condition $y(x_0) = \rho_0 = a_0, y'(x_0) = \rho_1 = a_1, y''(x_0) = \rho_2 = 2a_2$. then the ar 's and the tau parameters for various value of n is determined. Numerical experiments were given to illustrate the effectiveness of the method.

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INTRODUCTION

Tau method approximate solution of a given differential equation defined on a compact $[a,b]$ is obtained by adding to the right hand side of the equation a specific minimal polynomial perturbation term $H_n(x)$ which plays the role of a representation of zero in $[a,b]$ be elements of a given subspace of polynomials. The procedures based on the Tau method have been studied with the techniques being applied to more general equations as well as the nonlinear equations. (Yisa and Adeniyi, 2015). In this direction, Lanczos (1975) developed a modification based on the use of canonical polynomials. Also, two variants namely the differential and the integrated formulations were discussed by Ojo and Adeniyi (2012). However, Adeniyi (2008) based the error of Lanczos economization process to construct a polynomial error approximant of the error function $e_n(x)$ of the Lanczos Tau Method for ordinary differential equations, where he modified the approximant by perturbing some of the homogeneous conditions of $e_n(x)$. Yisa and Adeniyi (2012) developed a generalization of the canonical polynomials for over-determined order ordinary differential equations (ODEs). Moreover Issa *et al.* (2017) discussed the numerical method for solving non-linear ordinary differential equations based on the generalized tau

approximant. The error estimation of the tau method for both linear and non linear ordinary differential equations overdetermined and non overdetermined type were also examined and generalized with the use of Chebyshev polynomials as perturbed term.

Adeniyi and Ma'ali (2012) reported an error estimation of the integrated variant of the tau method for Initial Value Problems (IVPs) for the class of equations for which $m + s \leq 3$ where m and s are the order and number of over determination respectively. Badeggi *et al.* (2017) derived a generalized formula or recurrence relation for the integrated variant of the tau method which captures the general class of problems involving m -th order ordinary differential equations. Also Ajiya *et al.* (2018) investigated a generalized differential formulation of the Lanczos approximant of the error function for certain fourth order boundary value problems with first degree over-determination in ODEs which is based on a modification of the error of Lanczos economization process. Other numerical methods be found in (Peter, 2017, 2018, Peter *et al.* 2019, 2020, and 2021).

In this study the differential form of tau method and its error estimation for a class of third order differential equation is presented.



THE DIFFERENTIAL FORM OF TAU METHOD

Let us consider the m-th order linear differential system, that is

$$Ly_n(x) = \sum_{r=0}^m p_r(x)y_n^{(r)}(x) = f(x) + H_n(x) \quad (1)$$

$$L^*y_n(x_{rk}) = \sum_{r=0}^m a_{rk}y_n^{(r)}(x_{rk}) = a_k, \quad k = 1(1)m \quad (2)$$

The idea of Lanczos is to approximate the solution of the differential equation by an n-th degree polynomial function

$$y_n(x) = \sum_{r=0}^n a_r x^r, n < \infty \quad (3)$$

to give the exact solution of a perturbed equation obtained by adding to the perturbation term of the right hand side of the polynomial. The polynomial satisfies

$$Ly_n(x) = \sum_{r=0}^m P_r(x)y_r^{(r)}(x) = f(x) + H_n(x) \quad (4)$$

where

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x). \quad (5)$$

where $H_n(x)$ is taken as a linear combination of power of x multiplied by chebyshev polynomial. We equate the corresponding coefficients of x and also using the initial conditions, we then find the solution of the system of equation.

The Differential Form of the Tau method for a Class of Third Order Differential Equations

The third order initial value problem

$$y'''(x) + (P_{10} + P_{11}x + P_{12}x^2)y'(x) + (P_{00} + P_{01}x)y(x) = 0 \quad (6)$$

$$y(x_0) = \rho_0, y'(x_0) = \rho_1, y''(x_0) = \rho_2$$

is considered, where $P_{00}, P_{01}, P_{10}, P_{11}, P_{12}$ are given real constants.

Let

$$y_n(x) = \sum_{r=0}^n a_r x^r \quad (7)$$

be the approximate solution of (6). This satisfies the perturbed equation

$$Ly_n(x) = f(x) + H_n(x) \quad (8)$$

where the perturbation term $H_n(x)$ is

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \quad 0 \leq x \leq 1 \quad (9)$$

where s and m denote the number of over determination and the order of differential equation respectively.

$$T_r(x) = \cos\{r \cos^{-1} \left[\frac{2x-(b-a)}{b-a} \right]\} \quad (10)$$

is the r^{th} degree chebyshev polynomial in $[a,b]$. In this case $m = 3, s = 1$.

using (10), the chebychev polynomial in $[0,1]$ is

$$T_r(x) = \cos[r \cos^{-1}(2x - 1)] \quad (11)$$

and the recurrence relation is given by

$$T_{n+1}(x) = 2(2x - 1)T_n(x) - T_{n-1}(x) \quad (12)$$

Derivation of a nth degree Approximant

Consider the left hand side of (6), that is

$$Ly(x) \equiv \left(\frac{d^3}{dx^3} + (P_{10} + P_{11}x + P_{12}x^2) \frac{d}{dx} + (P_{00} + P_{01}x) \right) y(x)$$

Therefore

$$Ly_n(x) = \sum_{r=0}^n r(r-1)(r-2)a_r x^{r-3} + (P_{10} + P_{11}x + P_{12}x^2) \sum_{r=0}^n r a_r x^{r-1} + (P_{00} + P_{01}x) \sum_{r=0}^n a_r x^r \quad (13)$$

Thus



$$Ly_n(x) = \sum_{r=0}^n r(r-1)(r-2)a_r x^{r-3} + P_{10} \sum_{r=0}^n r a_r x^{r-1} + P_{11} \sum_{r=0}^n r a_r x^r + P_{12} \sum_{r=0}^n r a_r x^{r+1} + P_{00} \sum_{r=0}^n a_r x^r + P_{01} \sum_{r=0}^n a_r x^{r+1} \quad (14)$$

Putting $r = k + 3, k + 1, k, k - 1, k, k - 1$ into the first, second, third, fourth, fifth, and sixth term of the series respectively, we obtain

$$\begin{aligned} Ly_n(x) &= \sum_{r=0}^{n-3} (k+3)(k+2)(k+1)a_{k+3}x^k + P_{10} \sum_{r=0}^{n-1} (k+1)a_{k+1}x^k + P_{11} \sum_{r=0}^n k a_k x^k x + \\ &P_{12} \sum_{r=0}^{n+1} (k-1)a_{k-1}x^k + P_{00} \sum_{r=0}^n a_k x^k + P_{01} \sum_{r=0}^{n+1} a_{k-1}x^k \\ &= P_{10}(n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2}) + P_{11}(n a_n x^n + (n-1)a_{n-1}x^{n-1} + \\ &(n-2)a_{n-2}x^{n-2}) + P_{12}(n a_n x^{n+1} + (n-1)a_{n-1}x^n + (n-2)a_{n-2}x^{n-1} + \\ &(n-3)a_{n-3}x^{n-2}) + P_{00}(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2}) + P_{01}(a_n x^{n+1} + \\ &a_{n-1}x^n + a_{n-2}x^{n-1} + a_{n-3}x^{n-2}) + \sum_{0}^{n-3} \{P_{10}(k+1)a_{k+1} + P_{11}k a_k + \\ &P_{12}(k-1)a_{k-1} + P_{00}a_k + P_{01}a_{k-1} + (k+3)(k+2)(k+1)a_{k+3}\}x^k \end{aligned} \quad (15)$$

For the RHS, we have

$$\begin{aligned} Ly_n(x) &= \tau_4 T_{n-2}(x) + \tau_3 T_{n-1}(x) + \tau_2 T_n(x) + \tau_1 T_{n+1}(x) \\ &= \tau_4 \sum_{r=0}^{n-2} C_r^{(n-2)} x^r + \tau_3 \sum_{r=0}^{n-1} C_r^{(n-1)} x^r + \tau_2 \sum_{r=0}^n C_r^{(n)} x^r + \tau_1 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r \\ &= (\tau_4 C_{n-2}^{(n-2)} + \tau_3 C_{n-2}^{(n-1)} + \tau_2 C_{n-2}^{(n)} + \tau_1 C_{n-2}^{(n+1)})x^{n-2} + (\tau_3 C_{n-1}^{(n-1)} + \tau_2 C_{n-1}^{(n)} + \\ &\tau_1 C_{n-1}^{(n+1)})x^{n-1} + (\tau_2 C_n^{(n)} + \tau_1 C_n^{(n+1)})x^n + (\tau_1 C_{n+1}^{(n+1)})x^{n+1} + \sum_{0}^{n-3} (\tau_4 C_k^{(n-2)} + \\ &\tau_3 C_k^{(n-1)} + \tau_2 C_k^{(n)} + \tau_1 C_k^{(n+1)})x^k \end{aligned} \quad (16)$$

Equating corresponding coefficients give,

$$\begin{aligned} P_{10}(n-1)a_{n-1} + P_{11}(n-2)a_{n-2} + P_{12}(n-3)a_{n-3} + P_{00}a_{n-2} + P_{01}a_{n-3} &= \tau_4 C_{n-2}^{(n-2)} + \\ \tau_3 C_{n-2}^{(n-1)} + \tau_2 C_{n-2}^{(n)} + \tau_1 C_{n-2}^{(n+1)} \\ P_{10}n a_n + P_{11}(n-1)a_{n-1} + P_{12}(n-2)a_{n-2} + P_{00}a_{n-1} + P_{01}a_{n-2} &= \tau_3 C_{n-1}^{(n-1)} + \\ \tau_2 C_{n-1}^{(n)} + \tau_1 C_{n-1}^{(n+1)} \\ P_{11}n a_n + P_{12}(n-1)a_{n-1} + P_{00}a_n + P_{01}a_{n-1} &= \tau_2 C_n^{(n)} + \tau_1 C_n^{(n+1)} \\ P_{12}n a_n + P_{01}a_n &= \tau_1 C_{n+1}^{(n+1)} \\ P_{10}(k+1)a_{k+1} + P_{11}k a_k + P_{12}(k-1)a_{k-1} + P_{00}a_k + P_{01}a_{k-1} + \\ (k+3)(k+2)(k+1)a_{k+3} &= \tau_4 C_k^{(n-2)} + \tau_3 C_k^{(n-1)} + \tau_2 C_k^{(n)} + \tau_1 C_k^{(n+1)}; \quad k = 1, 2, 3, \dots, n-3 \end{aligned} \quad (17)$$

Using the initial conditions, we have

$$y(x_0) = \rho_0 = a_0, \quad y'(x_0) = \rho_1 = a_1, \quad y''(x_0) = \rho_2 = 2a_2$$

This system of equation is to be solved for the determination of a_r 's and the tau parameters for various values of n. consequently, from (7) we obtain the desired tau approximation $y_n(x)$ of $y(x)$.

ERROR ESTIMATION

We consider the error estimation for the class of problem

$$y'''(x) + (P_{10} + P_{11}x + P_{12}x^2)y'(x) + (P_{00} + P_{01}x)y(x) = 0 \quad (18)$$

$$y(x_0) = \rho_0, y'(x_0) = \rho_1, y''(x_0) = \rho_2$$

The error function

$$e_n(x) = y(x) - y_n(x) \quad (19)$$

Following Issa K. *et al.* (2017). The error approximant of the polynomial



$$(e_n(x))_{n+1} = \frac{\phi_n V_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \quad (20)$$

satisfies the perturbed error problem

$$L(e_n(x))_{n+1} = \tilde{H}_{n+1}(x) - H_n(x) \quad (21)$$

where

$$\tilde{H}_{n+1}(x) = \sum_{r=0}^{m+s-1} \tilde{\tau}_{m+s-r} T_{n-m+r+2}(x) \quad 0 \leq x \leq 1 \quad (22)$$

which implies

$$L(e_n(x))_{n+1} = \sum_{r=0}^{m+s-1} \tilde{\tau}_{m+s-r} T_{n-m+r+2} - \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1} \quad (23)$$

then

$$y'''(x) + (P_{10} + P_{11}x + P_{12}x^2)y'(x) + (P_{00} + P_{01}x)y(x) = \tilde{H}_{n+1}(x) - H_n(x) \quad (24)$$

$$\left[\frac{d^3}{dx^3} + (P_{10} + P_{11}x + P_{12}x^2) \frac{d}{dx} + (P_{00} + P_{01}x) \right] (e_n(x))_{n+1} = \tilde{H}_{n+1}(x) - H_n(x) \quad (25)$$

In this case, we consider

$$V_m(x) = (x - x_0)^m, \quad (m = 3)$$

$$(e_n(x))_{n+1} = \frac{\phi_n (x - x_0)^m T_{n-2}(x)}{C_{n-2}^{(n-2)}} = \frac{\phi_n x^3 T_{n-2}(x)}{C_{n-2}^{(n-2)}} \quad (26)$$

$$(e_n(x))_{n+1} = \theta \sum_{r=0}^{n-2} C_r^{(n-2)} x^{r+3} \quad (27)$$

where $\theta = \frac{\phi_n}{C_{n-2}^{(n-2)}}$

Now, From equation (25) we have

$$(e_n^{(3)}(x))_{n+1} + (P_{10} + P_{11}x + P_{12}x^2)(e_n^{(1)}(x))_{n+1} + (P_{00} + P_{01}x)(e_n(x))_{n+1} = \sum_{r=0}^3 \tilde{\tau}_{4-r} T_{n-1+r}(x) - \sum_{r=0}^3 \tau_{4-r} T_{n-2+r}(x) \quad (28)$$

From equation (27)

$$\begin{aligned} (e_n(x))_{n+1} &= \theta \sum_{r=0}^{n-2} C_r^{(n-2)} x^{r+3} \\ (e_n^{(1)}(x))_{n+1} &= \theta \sum_{r=0}^{n-2} (r+3) C_r^{(n-2)} x^{r+2} \\ (e_n^{(2)}(x))_{n+1} &= \theta \sum_{r=0}^{n-2} (r+3)(r+2) C_r^{(n-2)} x^{r+1} \\ (e_n^{(3)}(x))_{n+1} &= \theta \sum_{r=0}^{n-2} (r+3)(r+2)(r+1) C_r^{(n-2)} x^r \end{aligned} \quad (29)$$

Substituting equation (29) into (28), we have

$$\begin{aligned} &\theta \sum_{r=0}^{n-2} (r+3)(r+2)(r+1) C_r^{(n-2)} x^r + (P_{10} + P_{11}x + P_{12}x^2) \theta \sum_{r=0}^{n-2} (r+3) C_r^{(n-2)} x^{r+2} + \\ &(P_{00} + P_{01}x) \theta \sum_{r=0}^{n-2} C_r^{(n-2)} x^{r+3} = \tilde{\tau}_1 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r + \tilde{\tau}_2 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r + \tilde{\tau}_3 \sum_{r=0}^n C_r^{(n)} x^r + \\ &\tilde{\tau}_4 \sum_{r=0}^{n-1} C_r^{(n-1)} x^r - \tau_1 \sum_{r=0}^{n+1} C_r^{(n+1)} x^r - \tau_2 \sum_{r=0}^n C_r^{(n)} x^r - \tau_3 \sum_{r=0}^{n-1} C_r^{(n-1)} x^r - \tau_4 \sum_{r=0}^{n-2} C_r^{(n-2)} x^r \end{aligned} \quad (30)$$

Equating the corresponding power of $x^{n+2}, x^{n+1}, \dots, x^{n-2}$ gives



$$\begin{aligned} \tilde{\tau}_1 C_{n+2}^{(n+2)} &= \theta [P_{12}(n+1) + P_{01}] C_{n-2}^{(n-2)} \\ \tilde{\tau}_1 C_{n+1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n+1}^{(n+1)} &= \theta [[P_{11}(n+1) + P_{00}] C_{n-2}^{(n-2)} + [P_{12}(n) + P_{01}] C_{n-3}^{(n-2)}] \\ \tilde{\tau}_1 C_n^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_n^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_n^{(n)} &= \theta [[P_{10} + (n+1)] C_{n-2}^{(n-2)} + \\ & [P_{11}(n) + P_{00}] C_{n-2}^{(n-2)} + [P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)}] \\ \tilde{\tau}_1 C_{n-1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-1}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-1}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} &= \\ & \theta [[P_{10}(n)] C_{n-3}^{(n-2)} + [P_{11}(n-1) + P_{00}] C_{n-4}^{(n-2)} + [P_{12}(n-2) + P_{01}] C_{n-5}^{(n-2)}] \\ \tilde{\tau}_1 C_{n-2}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-2}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-2}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-2}^{(n-1)} - \tau_4 C_{n-2}^{(n-2)} &= \\ & \theta [[(n+1)(n)(n-1)] C_{n-2}^{(n-2)} + [P_{10}(n-1)] C_{n-4}^{(n-4)} + [P_{11}(n-2) + P_{00}] C_{n-5}^{(n-2)} + \\ & [P_{12}(n-3) + P_{01}] C_{n-6}^{(n-2)}] \end{aligned}$$

Making use of the following relation

$$\begin{aligned} C_k^{(k)} &= 2^{2k-1} \\ \frac{C_{k-1}^{(k)}}{C_k^{(k)}} &= -\frac{1}{2}k, \quad C_{n-1}^{(n)} = -n 2^{2n-2} \end{aligned}$$

Then the above system leads to

$$\begin{aligned} \tilde{\tau}_1 C_{n+2}^{(n+2)} &= \theta [P_{12}(n+1) + P_{01}] C_{n-2}^{(n-2)} \\ \tilde{\tau}_1 C_{n+2}^{(n+2)} &= \frac{\phi_n}{C_{n-2}^{(n-2)}} [P_{12}(n+1) + P_{01}] C_{n-2}^{(n-2)} \\ \tilde{\tau}_1 &= \frac{\phi_n}{C_{n+2}^{(n+2)}} [P_{12}(n+1) + P_{01}] \\ \text{Let } P_{12}(n+1) + P_{01} &= \alpha_1 \\ \tilde{\tau}_1 &= \frac{\alpha_1 \phi_n}{C_{n+2}^{(n+2)}} = \frac{\phi_n}{2^{2n+3}} \cdot \alpha_1 \\ \tilde{\tau}_1 C_{n+1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n+1}^{(n+1)} &= \theta [[P_{11}(n+1) + P_{00}] C_{n-2}^{(n-2)} + [P_{12}(n) + P_{01}] C_{n-3}^{(n-2)}] \end{aligned}$$



$$\begin{aligned}
 & \frac{\alpha_1 \phi_n}{C_{n+2}^{(n+2)}} \cdot C_{n+1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n+1}^{(n+1)} = \frac{\phi_n}{C_{n-2}^{(n-2)}} [P_{11}(n+1) + P_{00}] C_{n-2}^{(n-2)} + \\
 & [P_{12}(n) + P_{01}] C_{n-3}^{(n-2)} \\
 & - \frac{1}{2}(n+2)\alpha_1 \phi_n + 2^{2(n+1)-1}(\tilde{\tau}_2 - \tau_1) = \phi_n \left\{ [P_{11}(n+1) + P_{00}] - \frac{1}{2}(n-2)[P_{12}(n) + P_{01}] \right\} \\
 & 2^{2n+1}(\tilde{\tau}_2 - \tau_1) = \phi_n \left\{ [P_{11}(n+1) + P_{00}] - \frac{1}{2}(n-2)[P_{12}(n) + P_{01}] \right\} + \frac{1}{2}(n+2)\alpha_1 \phi_n \\
 & = \phi_n \left\{ [P_{11}(n+1) + P_{00}] - \frac{1}{2}(n-2)[P_{12}(n) + P_{01}] \right\} + \frac{1}{2}(n+2)\alpha_1 \\
 & = \frac{1}{2} \phi_n \{ [P_{11}(n+1) + P_{00}] - (n-2)[P_{12}(n) + P_{01}] + (n+2)\alpha_1 \} \\
 & 2^{2n+1}(\tilde{\tau}_2 - \tau_1) \cdot 2 = \phi_n \{ 2[P_{11}(n+1) + P_{00}] - (n-2)[P_{12}(n) + P_{01}] + (n+2)\alpha_1 \} \\
 & 2^{2n+1}(\tilde{\tau}_2 - \tau_1) = \phi_n \alpha_2 \\
 & \text{Where } \alpha_2 = 2[P_{11}(n+1) + P_{00}] - (n-2)[P_{12}(n) + P_{01}] + (n+2)\alpha_1 \\
 & \tilde{\tau}_2 - \tau_1 = \frac{\phi_n}{2^{2n+2}} \cdot \alpha_2 \\
 & \tilde{\tau}_1 C_n^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_n^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_n^{(n)} = \theta [P_{10} + (n+1)] C_{n-2}^{(n-2)} + \\
 & [P_{11}(n) + P_{00}] C_{n-2}^{(n-2)} + [P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)} \\
 & \Rightarrow \frac{\alpha_1 \phi_n}{C_{n+2}^{(n+2)}} \cdot C_n^{(n+2)} + \frac{\phi_n}{2^{2n+2}} \cdot \alpha_2 \cdot C_n^{(n+1)} + (\tau_3 - \tau_2) C_n^{(n)} = \frac{\phi_n}{C_{n-2}^{(n-2)}} \{ [P_{10} + (n+1)] + \\
 & C_{n-2}^{(n-2)} [P_{11}(n) + P_{00}] C_{n-2}^{(n-2)} + [P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)} \} \\
 & \Rightarrow (\tau_3 - \tau_2) C_n^{(n)} = \frac{\phi_n}{C_{n-2}^{(n-2)}} \{ [P_{10} + (n+1)] C_{n-2}^{(n-2)} + [P_{11}(n) + P_{00}] C_{n-2}^{(n-2)} \\
 & + [P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)} - \frac{\alpha_1 \phi_n}{C_{n+2}^{(n+2)}} \cdot C_n^{(n+2)} - \frac{\alpha_2 \phi_n}{2^{2n+2}} \cdot C_n^{(n+1)} \} \\
 & (\tau_3 - \tau_2) \cdot 2^{2n-1} = \phi_n [P_{10} + (n+1)] + \phi_n [P_{11}(n) + P_{00}] + \frac{\phi_n}{C_{n-2}^{(n-2)}} \cdot [P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)} - \\
 & \frac{\alpha_1 \phi_n}{C_{(n+2)n+2}} \cdot C_n^{(n+2)} - \frac{\alpha_2 \phi_n}{2^{2n+2}} \cdot C_n^{(n+2)}
 \end{aligned}$$

(31)



$$\begin{aligned}
 (\tau_3 - \tau_2) \cdot 2^{2n-1} &= \phi_n \left\{ [P_{10} + (n+1)] + [P_{11}(n) + P_{00}] + \frac{[P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)}}{C_{n-2}^{(n-2)}} - \right. \\
 &\quad \left. \frac{\alpha_1}{C_{n+2}^{(n+2)}} \cdot C_n^{(n+2)} - \frac{\alpha_2}{2^{2n+2}} \cdot C_n^{(n+1)} \right\} \\
 &= \phi_n \left\{ [P_{10} + (n+1)] + [P_{11}(n) + P_{00}] + \frac{[P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)}}{2^{2n-5}} - \frac{\alpha_1}{2^{2n+3}} \cdot C_n^{(n+2)} - \right. \\
 &\quad \left. \frac{\alpha_2}{2^{2n+2}} \cdot (n+1) 2^{2n} \right\} \\
 &= \phi_n \left\{ [P_{10} + (n+1)] + [P_{11}(n) + P_{00}] + \frac{[P_{12}(n-1) + P_{01}] C_{n-4}^{(n-2)}}{2^{2n-5}} - \frac{\alpha_1}{2^{2n+3}} \cdot C_n^{(n+2)} + \right. \\
 &\quad \left. 4\alpha_2(n+1) \right\} \\
 &= \phi_n \alpha_3 \\
 \tilde{\tau}_3 - \tau_2 &= \frac{\alpha_3 \phi_n}{2^{2n-1}} \\
 \tilde{\tau}_1 C_{n-1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-1}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-1}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} &= \\
 \theta [[P_{10}(n)] C_{n-3}^{(n-2)} + [P_{11}(n-1) + P_{00}] C_{n-4}^{(n-2)} + [P_{12}(n-2) + P_{01}] C_{n-5}^{(n-2)}] & \quad (32) \\
 \frac{\alpha_1 \phi_n}{C_{n+2}^{(n+2)}} \cdot C_{n-1}^{(n+2)} + \frac{\alpha_2 \phi_n}{2^{2n+2}} \cdot C_{n-1}^{(n+1)} + \frac{\alpha_3 \phi_n}{2^{2n-1}} \cdot C_{n-1}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} &= \\
 \frac{\phi_n}{C_{n-2}^{(n-2)}} \left\{ [P_{10}(n)] C_{n-3}^{(n-2)} + [P_{11}(n-1) + P_{00}] C_{n-4}^{(n-2)} + [P_{12}(n-2) + P_{01}] C_{n-5}^{(n-2)} \right\} \\
 (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} &= \frac{\phi_n}{C_{n-2}^{(n-2)}} P_{10}(n) C_{n-3}^{(n-2)} + \frac{\phi_n}{C_{n-2}^{(n-2)}} [P_{11}(n-1) + P_{00}] C_{n-4}^{(n-2)} + \\
 \frac{\phi_n}{C_{n-2}^{(n-2)}} [P_{12}(n-2) + P_{01}] C_{n-5}^{(n-2)} - \frac{\alpha_1 \phi_n}{C_{n+2}^{(n+2)}} \cdot C_{n-1}^{(n+2)} + \frac{\alpha_2 \phi_n}{2^{2n+2}} \cdot C_{n-1}^{(n+1)} + \\
 \frac{\alpha_3 \phi_n}{2^{2n-1}} \cdot C_{n-1}^{(n)} \\
 (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} &= -\frac{1}{2} (n-2) \phi_n P_{10}(n) + \frac{\phi_n [P_{11}(n-1) + P_{00}] C_{n-4}^{(n-2)}}{2^{2(n-2)-1}} + \\
 \frac{\phi_n [P_{12}(n-2) + P_{01}] C_{n-5}^{(n-2)}}{2^{2(n-2)-1}} - \frac{\alpha_1 \phi_n}{2^{2(2+2)-1}} \cdot C_{n-1}^{(n+2)} - \frac{\alpha_2 \phi_n}{2^{2n+2}} \cdot C_{n-1}^{(n+1)} - \\
 \frac{\alpha_3 \phi_n}{2^{2n-1}} \cdot -n 2^{2n-2}
 \end{aligned}$$



$$\begin{aligned}
 (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} &= \left\{ -\frac{1}{2}(n-2)P_{10}(n) + \frac{[P_{11}(n-1)+P_{00}]C_{n-4}^{(n-2)}}{2^{2n-5}} + \right. \\
 &\left. \frac{[P_{12}(n-2)+P_{01}]C_{n-5}^{(n-2)}}{2^{2n-5}} - \frac{\alpha_1 C_{n-1}^{(n+2)}}{2^{2n+3}} - \frac{\alpha_2}{2^{2n+2}} \cdot C_{n-1}^{(n+1)} + \frac{1}{2}n \cdot \alpha_3 \right\} \\
 \tilde{\tau}_4 - \tau_3 &= \frac{\phi_n}{2^{2n-3}} \cdot \alpha_4 \\
 \tilde{\tau}_1 C_{n-2}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-2}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-2}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-2}^{(n-1)} - \tau_4 C_{n-2}^{(n-2)} &= \\
 \theta \left[(n+1)(n)(n-1) C_{n-2}^{(n-2)} + [P_{10}(n-1)] C_{n-4}^{(n-2)} + [P_{11}(n-2) + P_{00}] C_{n-5}^{(n-2)} + \right. \\
 &\left. [P_{12}(n-3) + P_{01}] C_{n-6}^{(n-2)} \right] \\
 \frac{\alpha_1}{C_{n+2}^{(n+2)}} \cdot C_{n-2}^{(n+2)} + \frac{\phi_n}{2^{2n+2}} \cdot \alpha_2 C_{n-2}^{(n+1)} + \frac{\alpha_3}{2^{2n-1}} \cdot \phi_n \cdot C_{n-2}^{(n)} + \frac{\phi_n}{2^{2n-3}} \cdot \alpha_4 \cdot C_{n-2}^{(n-1)} - \\
 \tau_4 2^{2n-5} &= \frac{\phi_n}{C_{n-2}^{(n-2)}} \left\{ (n+1)(n)(n-1) C_{n-2}^{(n-2)} + [P_{10}(n-1)] C_{n-4}^{(n-2)} + [P_{11}(n-2) + P_{00}] \right. \\
 &\left. C_{n-5}^{(n-2)} + [P_{12}(n-3) + P_{01}] C_{n-6}^{(n-2)} \right\} - \tau_4 2^{2n-5} = \phi_n \\
 \frac{\phi_n}{C_{n+2}^{(n+2)}} \cdot \alpha_1 \cdot C_{n-2}^{(n+2)} + \frac{\phi_n}{2^{2n+2}} \cdot \alpha_2 C_{n-2}^{(n+1)} + \frac{\phi_n}{2^{2n-1}} \cdot \alpha_3 \cdot C_{n-2}^{(n)} + \\
 &\frac{\phi_n}{2^{2n-3}} \cdot \alpha_4 \cdot C_{n-2}^{(n-1)} - \\
 \tau_4 2^{2n-5} &= \phi_n \left\{ (n+1)(n)(n-1) + \frac{[P_{10}(n-1)]C_{n-4}^{(n-2)}}{2^{2n-5}} + \frac{[P_{11}(n-2)+P_{00}]C_{n-5}^{(n-2)}}{2^{2n-5}} + \right. \\
 &\left. \frac{[P_{12}(n-3)+P_{01}]C_{n-6}^{(n-2)}}{2^{2n-5}} - \frac{\alpha_1}{2^{2n+3}} \cdot C_{n-2}^{(n+2)} - \frac{\alpha_2}{2^{2n+2}} \cdot C_{n-2}^{(n+1)} - \right. \\
 &\left. \frac{\alpha_3}{2^{2n-1}} \cdot C_{n-2}^{(n)} + \alpha_4 \cdot \frac{1}{2}(n-1) \right\} \\
 \tau_4 \cdot 2^{2n-5} &= \phi_n \left\{ \frac{\alpha_1}{2^{2n+3}} \cdot C_{n-2}^{(n+2)} - \frac{\alpha_2}{2^{2n+2}} \cdot C_{n-2}^{(n+1)} + \frac{\alpha_3}{2^{2n-1}} \cdot C_{n-2}^{(n)} - \alpha_4 \cdot \frac{1}{2}(n-1) - \right. \\
 &\left. (n+1)(n)(n-1) - \frac{[P_{10}(n-1)]C_{n-4}^{(n-2)}}{2^{2n-5}} + \frac{[P_{11}(n-2)+P_{00}]C_{n-5}^{(n-2)}}{2^{2n-5}} - \right. \\
 &\left. \frac{[P_{12}(n-3)+P_{01}]C_{n-6}^{(n-2)}}{2^{2n-5}} \right\}
 \end{aligned}$$

(33)



where

$$\alpha_5 = \frac{\alpha_1}{2^{2n+3}} \cdot C_{n-2}^{(n+2)} - \frac{\alpha_2}{2^{2n+2}} \cdot C_{n-2}^{(n+1)} + \frac{\alpha_3}{2^{2n-1}} \cdot C_{n-2}^{(n)} - \alpha_4 \cdot \frac{1}{2} (n-1) -$$

$$(n+1)(n)(n-1) - \frac{[P_{10}(n-1)] C_{n-4}^{(n-2)}}{2^{2n-5}} + \frac{[P_{11}(n-2)+P_{00}]}{2^{2n-5}} C_{n-5}^{(n-2)} -$$

$$\frac{[P_{12}(n-3)+P_{01}] C_{n-6}^{(n-2)}}{2^{2n-5}}$$

$$\alpha_4 = -\frac{1}{2} (n-2) P_{10}(n) + \frac{[P_{11}(n-1)+P_{00}] C_{n-4}^{(n-2)}}{2^{2n-5}} + \frac{[P_{12}(n-2)+P_{01}] C_{n-5}^{(n-2)}}{2^{2n-5}} -$$

$$\frac{\alpha_1 C_{n-1}^{(n+2)}}{2^{2n+3}} - \frac{\alpha_2}{2^{2n+2}} \cdot C_{n-1}^{(n+1)} + \frac{1}{2} n \cdot \alpha_3$$

$$\alpha_3 = [P_{10} + (n+1)] + [P_{11}(n) + P_{00}] + \frac{[P_{12}(n-1)+P_{01}] C_{n-4}^{(n-2)}}{2^{2n-5}} - \frac{\alpha_1}{2^{2n+3}} \cdot C_n^{(n+2)} +$$

$$4\alpha_2(n+1)$$

$$\alpha_2 = 2[P_{11}(n+1) + P_{00}] - (n-2)[P_{12}(n) + P_{01}] + (n+2)\alpha_1$$

$$\alpha_1 = P_{12}(n+1) + P_{01}$$

NUMERICAL EXAMPLES AND RESULTS

Numerical Examples

We consider here three selected numerical problems for experimentation with our results of preceding section, using different values of $P_{00}, P_{01}, P_{10}, P_{11}, P_{12}$

Example 1

$$y'''(x) + (1+x^2)y'(x) - xy(x) = 0 \quad 0 \leq x \leq 1$$

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1$$

For this example, $P_{00} = 0, P_{01} = -1, P_{10} = 1, P_{11} = 0, P_{12} = 1$ and from the initial condition we have

$$a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}$$

Example 2

$$y'''(x) + (1-x+2x^2)y'(x) + (1+2x)y(x) = 0$$

$$y(0) = 0, y'(0) = 1, y''(0) = 0$$

For this example, $P_{00} = 1, P_{01} = 2, P_{10} = 1, P_{11} = -1, P_{12} = 2$ and from the initial condition we have

$$a_0 = 0, a_1 = 1, a_2 = 0$$

Numerical Results

The numerical results for these examples are presented in the tables below

Table 1: Estimated error for problem 1

x	$ y_5(x) - y_4(x) $	$ y_6(x) - y_5(x) $	$ y_7(x) - y_6(x) $	$ y_8(x) - y_7(x) $	$ y_9(x) - y_8(x) $	$ y_{10}(x) - y_9(x) $
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	2.41×10^{-6}	7.47×10^{-7}	1.43×10^{-7}	3.90×10^{-8}	6.00×10^{-9}	0.00
0.2	3.67×10^{-5}	1.07×10^{-5}	1.84×10^{-6}	4.22×10^{-7}	4.90×10^{-8}	0.00×10^{-9}
0.3	1.77×10^{-4}	4.78×10^{-5}	7.47×10^{-6}	1.53×10^{-6}	1.60×10^{-7}	1.00×10^{-9}
0.4	5.29×10^{-4}	1.33×10^{-5}	1.89×10^{-5}	3.48×10^{-6}	3.37×10^{-7}	1.00×10^{-8}
0.5	1.22×10^{-3}	2.85×10^{-4}	3.68×10^{-5}	6.30×10^{-6}	6.04×10^{-7}	1.60×10^{-8}
0.6	2.39×10^{-3}	5.16×10^{-4}	6.13×10^{-5}	1.01×10^{-5}	9.99×10^{-7}	2.70×10^{-8}

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0.7	4.15×10^{-3}	8.32×10^{-4}	9.23×10^{-5}	1.52×10^{-5}	1.57×10^{-6}	4.60×10^{-8}
0.8	6.62×10^{-3}	1.23×10^{-3}	1.30×10^{-4}	2.19×10^{-5}	2.32×10^{-6}	6.40×10^{-8}
0.9	9.86×10^{-3}	1.71×10^{-3}	1.76×10^{-4}	3.06×10^{-5}	2.32×10^{-6}	8.90×10^{-8}
1.0	1.39×10^{-2}	2.26×10^{-3}	2.33×10^{-4}	4.11×10^{-5}	4.37×10^{-6}	1.20×10^{-7}

Table 2: Error estimate for problem 1

N/Error	Error estimate
6	4.00×10^{-7}
7	4.09×10^{-8}
8	2.62×10^{-9}
9	4.61×10^{-11}
10	1.09×10^{-11}

Table 3: Estimated error for problem 2

x	$ y_5(x) - y_4(x) $	$ y_6(x) - y_5(x) $	$ y_7(x) - y_6(x) $	$ y_8(x) - y_7(x) $	$ y_9(x) - y_8(x) $	$ y_{10}(x) - y_9(x) $
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	1.83×10^{-4}	5.38×10^{-6}	2.21×10^{-6}	4.85×10^{-7}	2.56×10^{-8}	1.12×10^{-8}
0.2	1.48×10^{-3}	7.68×10^{-5}	2.86×10^{-5}	5.55×10^{-6}	2.53×10^{-7}	9.55×10^{-8}
0.3	5.05×10^{-3}	3.45×10^{-4}	1.16×10^{-4}	2.00×10^{-5}	8.14×10^{-7}	2.81×10^{-7}
0.4	1.20×10^{-2}	9.63×10^{-4}	2.94×10^{-4}	4.57×10^{-5}	1.73×10^{-6}	5.99×10^{-7}
0.5	2.36×10^{-2}	2.06×10^{-3}	5.74×10^{-4}	8.28×10^{-5}	3.10×10^{-6}	1.13×10^{-6}
0.6	4.06×10^{-2}	3.73×10^{-3}	9.56×10^{-4}	1.32×10^{-4}	5.14×10^{-6}	1.95×10^{-6}
0.7	6.41×10^{-2}	6.01×10^{-3}	1.44×10^{-3}	1.99×10^{-4}	8.05×10^{-6}	3.07×10^{-6}
0.8	9.46×10^{-2}	8.87×10^{-3}	2.02×10^{-3}	2.86×10^{-4}	1.19×10^{-5}	4.49×10^{-6}
0.9	1.11×10^{-1}	1.23×10^{-2}	2.73×10^{-3}	3.98×10^{-4}	1.66×10^{-5}	6.25×10^{-6}
1.0	1.78×10^{-1}	1.61×10^{-2}	3.58×10^{-3}	5.33×10^{-4}	2.22×10^{-5}	8.39×10^{-6}

Table 4: Error estimate for problem 2

N/Error	Error estimate
6	3.05×10^{-6}
7	2.71×10^{-7}
8	6.99×10^{-9}
9	1.72×10^{-9}
10	1.71×10^{-10}

The estimated error is defined as
 Estimated error = $|y_{N+1}(x) - y_N(x)|$, $N = 4(1)10$. The error estimate is given by 35

CONCLUSION

The derivation of the tau method and its error estimate for a special class of third order differential equation has been presented. The subject of this study has been applied to two specific problems for which the true solutions are

unknown. Consequently, the estimated error and error estimate were determined and both were found to improve, tending to converge to the expected true error, as N increases.



REFERENCES

- Adeniyi, R. B. (2008). An improved error estimation of the Tau method for boundary value problems in ordinary differential equations. *Research Journal of Applied Sciences* 3(6):456-464.
- Adeniyi R. B. (1991). On the Tau method for numerical solution of ordinary differential equations. *Doctoral Thesis (unpublished), University of Ilorin, Ilorin, Nigeria.*
- Adeniyi, R. B and Ma'ali, A. I. (2012). An error estimation of the Tau method for some class of ordinary differential equations. *Journal of Mathematics.* 2:32-40.
- Ajiya, Y., Wakili, A., Awwal, A. M., Abubakar, A. and Audu, A. (2018). Error estimation of the differential Tau method for certain fourth order boundary value problems. *Journal of Scientific Research and Studies.* 5(1):15-20.
- Badeggi, A.Y., Ma'ali, A.I., Abubakar, A., Abubakar, A.W. and Mohammed, U. (2017). A generalized formulation for integrated variant of Tau method for overdetermined m-th order ordinary differential equations. *Nigerian Research Journal of Engineering and Environmental Sciences.* 2(1):203-214.
- Issa, K., Adeniyi, R. B. and Yisa, B. M. (2017). Generalized error estimation of the Tau method in ordinary differential equations. *Journal of the Nigerian Mathematical Society.* 36:113-137.
- Lanczos, C. (1938). Trigonometric interpolation of empirical and analytic functions. *Journal of Mathematical Physics.* 17:123-199.
- Lanczos, C. (1956). *Applied Analysis*, Prentice Hall, New Jersey.
- Lanczos, C. (1975). *Step by step Tau Method-Part I. Piecewise polynomial approximations*, Imperial College, University of London, London.
- Ma'ali, A. I. and Adeniyi, R. B. (2012). On the integrated formulation of the Tau method involving at most two Tau parameters for IVPs in ODEs. *Journal of Mathematics.* 2:23-31.
- Ojo, V. O. and Adeniyi, R. B. (2012). The differential form of the Tau method and its error estimate for third order non-overdetermined differential equations. *General Mathematics Notes.* 11(1):41-49.
- Ortiz E. L. (1969). The tau method, *SIAM Journal of Numerical Analysis.* 6: 480-492.
- Peter, O. J. (2020.) *Transmission Dynamics of Fractional Order Brucellosis Model Using Caputo-Fabrizio Operator*, *Int. J. Differ. Equ.* 2020 (2020), 2791380.
- Peter, O. J and Awoniran, (2018) A.F. Homotopy perturbation method for solving sir infectious disease model by incorporating vaccination. *Pac. J. Sci. Technol.* 19(1), 133-140.
- Peter, O. J and Ibrahim M. O. (2017) *Application of Differential Transform Method in Solving Typhoid Fever Model*. *Int. J. Math. Anal. Optim., Theory Appl.* 2017 250-260.
- Peter, O. J. Shaikh, A.S. Ibrahim, M. O. Nisar, K.S. Baleanu, D., Khan, I., and Abioye, A.I. (2021) *Analysis and Dynamics of Fractional Order Mathematical Model of COVID-19 in Nigeria Using Atangana-Baleanu Operator*. *Computers, Materials and Continua.* 66(2), 1823-1848(2021).
- Peter, O. J., Abayomi, A. A., Adebisi, A. F., Ayoola, T. A. and Bitrus, S. (2018) *Solutions of the SIR-B Cholera Model Using Homotopy Analysis Method*. *Futo Journal Series (FUTOJNLS)* 4(2), 168 - 176
- Oyedepo, T., Adebisi, A.F. Tayo, R. M. Adedeji, J. A. Peter, O.J. (2021.) *Perturbed least squares technique for solving volterra fractional integro-differential equations based on constructed orthogonal polynomials*. *Journal of Mathematical and Computer Science.* 11, 203- 218
- Yisa, B. M. and Adeniyi, R. B. (2015). On generalization of the error and error estimation process of Ortiz's recursive formulation of the Tau method. *Journal of Nigerian Mathematical Society.* 34:70-82.