

## ON THE EXISTENCE OF ALGEBRAIC LIMIT CYCLE OF QUADRATIC SYSTEMS

By

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*The invariant algebraic curves of quadratic polynomial differential system using Darboux integral were determined. Also the existence of algebraic limit cycles based on the existing theorems; that provides necessary and sufficient conditions on the cofactors of the invariant algebraic curves for the existence of algebraic limit cycles of quadratic differential system were established. Concrete examples were given to illustrate the algebraic limit cycle for quadratic differential system.*

**Keywords:** Algebraic Limit Cycle, Quadratic Vector Field, Invariant Algebraic Curve, Darboux First Integral

**INTRODUCTION**

Existence of algebraic limit cycle is very significant in the qualitative theory of differential equation and it usually represents an important phenomenon whenever it occurred. In 1900 Hilbert in the second part of its 16<sup>th</sup> problem proposed to find an estimation of the uniform upper bound for the number of limit cycle of all polynomial vector field of a given degree and also their distribution or configuration in the plane. This has been one of the main problems in the qualitative theory of planar differential system in XX century and it remains open even for quadratic polynomial vector field. Poincaré (1891) defines a limit cycle of a polynomial vector field  $X$  as an isolated periodic orbit in the set of all periodic orbits of  $X$ , that is periodic orbit such that there exists an annular neighborhood without any other closed orbit. Llibre (2008) defined an algebraic limit cycle of polynomial vector field  $X$  as an oval of an irreducible invariant curve of equation (I). That is, a limit cycle of  $X$  contained in some irreducible invariant algebraic curve  $f(x, y) = 0$  of equation (I), and the degree of the algebraic limit cycle is the degree of  $f(x, y)$ . Existence and determination of algebraic

limit in the plane constitutes a difficult problem in the theory of differential equations. Many mathematicians worked on the existence and uniqueness of algebraic limit cycle of equation (I). Yablonskii (1996) found the families of algebraic limit cycle of degree two and four. It has been proved by Evdokimenco (1970) that there are no algebraic limit cycles of degree three. The existence of algebraic limit cycle in generalized Abel equation has recently been studied by Alwash (2007). Llibre and Swizez (2006) studied and classified the quadratic systems that admit the existence of algebraic limit cycles based on geometric properties of the embedding the system in the Poincaré compactification of  $\mathbb{R}^2$ , and the families are classified as: Qin-Yun's system, Yablonskii's system, Filipov's system, Chavarriga's system, Soroll's system and Christopher's system.

Darboux (1878) found a connection between algebraic geometry and the existence of limit cycle of polynomial system. Therefore, the existence of invariant algebraic curves is the central objectives in the theory of integrability of polynomial system in  $\mathbb{R}^2$  which has been considered by several mathematicians, (Druzhkova (1968), Tsyvintser (2001), Chavarriga (2004)).

A real planar polynomial differential system is a differential system of the form:

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y) \quad \dots (1)$$

Where  $P$  and  $Q$  are real polynomials in two variables  $x$  and  $y$ . The degree  $n$  of the polynomial system given in equation (1) is the maximum degree of the

polynomials  $P$  and  $Q$ . Associated to the polynomial system (1) there is a real polynomial vector field

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}, \text{ or simply } X = (P, Q) \quad \dots (2)$$

Let  $f = f(x, y)$  be a real polynomial in the variables  $x$  and  $y$ . The algebraic curve  $f(x, y) = 0$  of  $\mathbb{R}^2$  is an invariant algebraic curve of the vector field  $X$  if for some polynomial  $k \in \mathbb{R}^2$  such that

$$Xf = P(x, y) \frac{\partial f}{\partial x} + Q(x, y) \frac{\partial f}{\partial y} = kf \quad \dots (3)$$

The polynomial  $k$  is called the cofactor of the invariant algebraic curve  $f(x, y) = 0$ . An invariant algebraic curve  $f(x, y) = 0$  is irreducible if the polynomial  $k$  is irreducible. If the degree of the

polynomial system given in equation (1) is  $m$  then the degree of the cofactor of invariant algebraic curve of the system is at most  $m - 1$ .

## 2.0 Darboux First Integral

Darboux (1878) showed that if polynomial vector field of degree  $m$  is Darboux integrable then it has a Darboux first integral of the form:

$$H(x, y) = \prod_{i=1}^n f_i^{\lambda_i}(x, y) \quad \dots (4)$$

Where  $\lambda_i \in \mathbb{R}$  and  $f_i(x, y) = 0$  are the invariant algebraic curves of equation (1). The Darboux first integral in equation (4) is a function that can be expressed as a product of invariant algebraic curves. Therefore; the invariant algebraic curves play a significant role in the construction the integrating factor

of Darboux type. Therefore, existences of Darboux first integral and invariant algebraic curves are among the mechanisms that determine the existence of algebraic limit cycle of a quadratic vector field.

Mohammadi and Shariati (2006) stated that associated to each Darboux first integral  $H(x, y)$  there is an integrating factor  $R(x, y)$  that Satisfies  $\frac{\partial H(x,y)}{\partial x} = -R(x, y)Q(x, y)$  such that

$$H(x, y) = \prod_{i=1}^q f_i^{\lambda_i}(x, y) = \int R(x, y)P(x, y) dy + f(x) \quad \dots (5)$$

The integrating factor  $R(x, y)$  is obtained as follows:

$$(i) R(x, y) = e^{-\int \left( \frac{\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}}{P(x,y)} \right) dx} \quad \text{or} \quad (ii) R(x, y) = e^{-\int \left( \frac{\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}}{Q(x,y)} \right) dy}$$

Which also depend on the real planer differential system given in equation (1)

### 3.0 Existence of Algebraic Limit Cycle

The method of Darboux integral is utilized to establish a result (proposition) by using some known results, i.e. Christopher and Moulin (2005), Llibre and

Swizez (2006) etc to obtain the existence of algebraic limit cycle of quadratic system of the form:

$$\dot{x} = P(x, y) = \sum_{i,j=0}^2 p_{i,j} x^i y^j, \quad \dot{y} = Q(x, y) = \sum_{i,j=0}^2 q_{i,j} x^i y^j \quad \dots (6)$$

Several examples are also given to illustrate how this proposition can be easily employed to establish the existence of algebraic limit cycle of quadratic system.

#### Theorem 1.0

Let  $\gamma$  be a limit cycle of the quadratic system given in equation (1) contained in the invariant algebraic curve  $f(x, y) = 0$ . Then  $\gamma$  must intersect the line  $k(x, y) = 0$ .

Proof: [see Christopher and Moulin (2005)]

#### Theorem 1.1

Suppose that system given in equation (1) has an invariant algebraic curve  $f(x, y) = 0$  and a smooth non constant function  $H$  defined in the complement of the set  $f(x, y) = 0$  satisfying either:

- (i)  $H$  has a fixed sign except for a finite set of points.
- (ii)  $\dot{H} = 0$

Then all the limit cycles of (16) are contained in the set  $f(x, y) = 0$ .

Proof: [see Christopher and Moulin (2005)]

**Theorem 1.2**

Let  $f(x, y) = 0$  be an invariant algebraic curve for system given in equation (1) with cofactor  $k(x, y)$  and degree of  $f(x, y) > 1$ . If the straight line  $k(x, y) = 0$  is isoclines of system of equation (1) then all the limit cycles of equation (1) are contained in the set  $f(x, y) = 0$ .

**Proof:**[see Llibre and Swizez (2006)]

**Proposition 1.0:** If a quadratic system has an algebraic limit cycle then it is regular.

**Proof:**

Consider a quadratic system of the form:

$$\dot{x} = P(x, y) = \sum_{i,j=0}^2 p_{i,j} x^i y^j, \quad \dot{y} = Q(x, y) = \sum_{i,j=0}^2 q_{i,j} x^i y^j \quad \dots (7)$$

The algebraic curve  $f(x, y) = 0$  is an invariant algebraic curve of equation (1) if there exists a polynomial  $k(x, y) \in R[x, y]$  satisfying:

$$P(x, y) \frac{\partial f(x, y)}{\partial x} + Q(x, y) \frac{\partial f(x, y)}{\partial y} = k(x, y) f(x, y) \quad \dots (8)$$

The polynomial  $k(x, y)$  is called a factor of the curve  $f(x, y) = 0$ . Therefore, in this case since we are considering quadratic systems then the degree of the cofactor  $k(x, y)$  is at most one. A quadratic system given in equation (8) with invariant algebraic curve  $f(x, y) = 0$  is said to be regular if the cofactor  $k(x, y)$  of the invariant algebraic curves is non-constant. That is  $k(x, y) = Lx + My + N$  where  $L, M, N \in \mathbb{R}$  [Chavarriga, J. (2001)]. From theorem 1.0 we already known that if the cofactor  $k(x, y)$  is non constant then all the limit cycles

contained in the set  $f(x, y) = 0$  must intersect line  $k(x, y) = 0$ . Now if the set  $\{(x, y) | f(x, y) \neq 0\}$  then let  $H(x, y) = \ln|f(x, y)|$ . Llibre and Swizez (2006) deduce that there are no limits cycles apart from the ones contained in the set  $f(x, y) = 0$ . now suppose that  $k$  is non constant and  $\gamma$  is a limit cycle not intersecting  $k(x, y) = 0$ . Then  $\dot{H}/\gamma$  has a fixed sign and we have a contradiction using the arguments from the proof of theorem 1.1 therefore, the equation (8) is regular and hence the proof.

**3.1 Examples**

In this section we are going to consider some examples to illustrate the theorems and proposition considered earlier.

Example 1

$$\begin{aligned} \dot{x} &= P(x, y) = y \\ \dot{y} &= Q(x, y) = -(x + x^2 + y^2) \end{aligned}$$

The integrating factor

$$R(x, y) = e^{-\int \left( \frac{\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}}{P(x,y)} \right) dx} = e^{\int 2\partial x} = e^{2x}$$

The Darboux integral

$$H(x, y) = \prod_{i=0}^q f_i^{\lambda_i}(x, y) = \int R(x, y)P(x, y) dy + f(x) = \int ye^{2x} dy + f(x) \\ = \frac{y^2 + e^{2x}}{2} + f(x)$$

Satisfying

$$\frac{\partial H(x, y)}{\partial x} = y^2 e^{2x} + f'(x) - R(x, y)Q(x, y) = e^{2x}(x + x^2 + y^2)$$

$\Rightarrow f(x) = \int e^{2x}(x^2 + x) dx = e^{2x}(x + x^2 + y^2) \Rightarrow H(x, y) = \prod_{i=1}^2 f_i^{\lambda_i}(x, y) = \frac{e^{2x}}{2}(x^2 + y^2) = \left(\frac{e^x}{\sqrt{2}}\right)^2 (x^2 + y^2)$  with  $\lambda_1 = 2, \lambda_2 = 1$ . Now,  $f_1(x, y) = \frac{e^x}{\sqrt{2}}, f_2(x, y) = (x^2 + y^2)$  are the invariant algebraic curves and their cofactors are  $k_1(x, y) = y, k_2(x, y) = 2y$  respectively.

Since both  $f_1(x, y)$  and  $f_2(x, y)$  are irreducible invariant algebraic curves cofactors with non constants cofactors  $k_1(x, y)$  and  $k_2(x, y)$  respectively, then Example 2

from proposition.1 the quadratic system in example 1 is regular and hence algebraic limit cycle exist.

$$\dot{x} = P(x, y) = 2xy$$

$$\dot{y} = Q(x, y) = y^2 - x^2$$

The integrating factor

$$R(x, y) = e^{-\int \left( \frac{\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}}{P(x,y)} \right) dx} = e^{-2 \int \frac{1}{x} dx} = \frac{1}{x^2}$$

Darboux integral given by

$$H(x, y) = \int R(x, y)P(x, y)dy + f(x) = \frac{2}{x} \int y dy + f(x) = \frac{y^2}{x} + f(x)$$

Satisfying

$$\frac{\partial H}{\partial x} = -\frac{y^2}{x^2} + f'(x) = -R(x, y)Q(x, y) = -\frac{1}{x^2}(y^2 - x^2) \Rightarrow f'(x) = 1 \text{ and } f(x) = x$$

$$\Rightarrow H(x, y) = \prod_{i=1}^2 f_i^{\lambda_i} = (x)^{-1} (x^2 + y^2) \text{ with } \lambda_1 = -1, \lambda_2 = 1,$$

Now,  $f_1(x, y) = x$ ,  $f_2(x, y) = (x^2 + y^2)$  are the invariant algebraic curves and their cofactors are  $k_1(x, y) = 2y$ ,  $k_2(x, y) = 2x$  respectively.

Also from proposition 1.0 the quadratic system in example 2 is regular since  $k_1(x, y)$  and  $k_2(x, y)$  are both non constant we can conclude that the limit cycle

exist since the algebraic curves  $f_1(x, y)$  and  $f_2(x, y)$  are all irreducible

Example 3

$$\dot{x} = P(x, y) = y$$

$$\dot{y} = Q(x, y) = x^2 + y^2 - x$$

The integrating factor

$$R(x, y) = e^{-\int \left( \frac{\frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y}}{P(x, y)} \right) dx} = e^{-\int 2 dx} = e^{-2x}$$

$$\Rightarrow H(x, y) = \int R(x, y) P(x, y) dy + f(x) = \int ye^{-2x} dy + f(x) = \frac{y^2}{2} e^{-2x} + f(x)$$

Satisfying

$$\begin{aligned} \frac{\partial H}{\partial x} &= -y^2 e^{-2x} + f'(x) = -RQ = -e^{-2x}(x^2 + y^2 - x) \Rightarrow f'(x) = -e^{-2x}(x^2 - x), f(x) \\ &= \frac{x^2 e^{-2x}}{2} \end{aligned}$$

$$\Rightarrow H(x, y) = \prod_{i=1}^2 f_i^{\lambda_i} = \left( \frac{e^{-x}}{\sqrt{2}} \right)^2 (x^2 + y^2) \text{ with } \lambda_1 = 2, \lambda_2 = 1,$$

Now  $f_1(x, y) = \frac{e^{-x}}{\sqrt{2}}$ ,  $f_2(x, y) = (x^2 + y^2)$  are the invariant algebraic curves and their cofactors are  $k_1(x, y) = -y$ ,  $k_2(x, y) = 2y$  respectively

Since both  $f_1(x, y)$  and  $f_2(x, y)$  are irreducible invariant algebraic curves cofactors with non constants cofactors  $k_1(x, y)$  and  $k_2(x, y)$  respectively, then

from proposition 1.0 the quadratic system in example 3 is also regular hence the algebraic limit cycle exist.

## CONCLUSION

Therefore, in this paper the existence of algebraic limits cycle of quadratic differential system was established based on the existing theorems that provide necessary and sufficient conditions on the

cofactors of invariant algebraic curves of quadratic system. Examples were given to illustrate the existence or otherwise of algebraic limit cycle of quadratic system using the Darboux integral.

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