



A 3-Step Single Algorithm Developed for Direct solution of Linear and Nonlinear First, Second, Third, Fourth and Fifth Orders of Ordinary Differential Equations

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ABSTRACT

A single algorithm has been developed to efficiently solve ordinary differential equations (ODEs) ranging from first to fifth order. This method utilizes interpolation and collocation techniques, with a power series function as its foundation. The algorithm accurately places derivatives at grid and off-grid points, except for the end points, and has undergone thorough examination of its fundamental properties. Results indicate that this novel approach exhibits superior accuracy when compared to existing methods for solving ODEs of these orders. Moreover, it eliminates the need for developing separate methods for each of the specified five categories of ODEs.

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INTRODUCTION

According to Jain (1978), to obtain accurate numerical solutions to differential equations governing physical systems has always been an important problem with scientists and engineers.

These differential equations basically fall into two classes, ordinary and partial, depending on the number of independent variables present in the differential equations: One for ordinary and more than one for partial. Many problems in real life situation, science and engineering and even in humanities can be formulated as mathematical models in which numerical analysis can render solution to. These models most often cannot be properly handled in their natural forms by mere analytical methods

because of their complex nature. The task then becomes tremendous when handling such problems. It is in this view that numerical methods have been found to be inevitable. This numerical method uses the available discrete numerical integration algorithms in which the numerical approximations are at some specified points in the interval of integration. Numerical evaluation of certain differential equations can be done easily even when the analytic solutions are available.

The numerical solution of First, Second, Third, Fourth and Fifth orders initial value problems (IVPs) of Ordinary Differential Equations (ODEs) directly using a single Linear Multistep Hybrid Block Method (LMHBMs) is our focus in this paper. These ODEs are generally written in the following form:

$$\left. \begin{aligned} y' &= f(x, y), y(x_0) = y_0 \\ y'' &= f(x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0 \\ y''' &= f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \\ y^{iv} &= f(x, y, y', y'', y'''), y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0, y'''(x_0) = y'''_0 \\ y^v &= f(x, y, y', y'', y''', y^{iv}), y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0, y'''(x_0) = y'''_0, y^{iv}(x_0) = y^{iv}_0 \end{aligned} \right\} \quad (1.1)$$

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Numerous numerical methods for solving these equations separately have been carried out by many authors such as Joshua (2021,2022), Faruk(2023), Morteza(2023), Muideen(2021,2023), Friday(2023), Dauda (2022), Victor(2021), Roseline(2021), Omar and Kuboye (2015), Ezekiel (2018), Abolarin (2020), and Raymond *et al.* (2021) respectively derived hybrid block linear multistep method for the direct solution of general second, third and fourth IVPs of ODEs. Generalized hybrid technique for the solution of second order ODEs directly was also done by Kuboye et al. (2018). The methods developed by all these researchers are effective and efficient in terms of accuracy but they are limited to or developed to solve only one of the ODEs in (1.1). However, Adeyefa and Kuboye (2020) derived a new numerical model suitable for solving simultaneously second and third order ODEs, John (2022) developed a single numerical algorithm for solving first and second orders ODEs. Again, Olusola and Bamikole (2022) formulated a new hybrid method for direct numerical solution of nonlinear second, third and fourth orders of ordinary differential equations. The aforementioned researchers did not look at

the possibility of harmonizing different categories of ODEs- starting from first, second, third and fourth orders of ODEs and developing a hybrid block method capable of providing approximate solution to the four categories of differential equations simultaneously. Therefore, this new unified numerical method with additional fifth order is formulated to solve equation (1.1). Hence, this new method rises above the setback in developing numerical methods for solving a single class of ODEs by proposing a hybrid block method that gives solution to ODEs of five different categories of order: One, Two, Three, four and five. Therefore, this proposed approach will also eliminate the burden of developing a separate method of the equations in (1.1).

METHODOLOGY

In this segment, we formulate a hybrid block linear multistep method for solving (1.1) using a combination of collocation and interpolation

Our approach involves representing the approximate solution to the targeted ODEs using power series as basis function of the form:

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^j \tag{2.1}$$

where a_j s are unknown coefficients to be determined, t and m are number of interpolation and collocation points respectively. The first, second, third, fourth and fifth derivatives of (2.1) are given by

$$y'(x) = \sum_{j=1}^{t+m-1} j a_j x^{j-1} \tag{2.2}$$

$$y''(x) = \sum_{j=2}^{t+m-1} j(j-1) a_j x^{j-2} \tag{2.3}$$

$$y'''(x) = \sum_{j=3}^{t+m-1} j(j-1)(j-2) a_j x^{j-3} \tag{2.4}$$

$$y^{iv}(x) = \sum_{j=4}^{t+m-1} j(j-1)(j-2)(j-3) a_j x^{j-4} \tag{2.5}$$

$$y^v(x) = \sum_{j=5}^{t+m-1} j(j-1)(j-2)(j-3)(j-4) a_j x^{j-5} \tag{2.6}$$

Interpolating (2.1) at $x = x_{n+2}$ and collocating equations (2.2) to (2.5) at $x = x_{n+3}$ and also collocating (2.6) at $x = x_n, x_{n+\frac{1}{8}}, x_{n+1}, x_{n+\frac{9}{8}}, x_{n+2}, x_{n+\frac{17}{8}}, x_{n+3}$ gives a system of nonlinear equation of the form:

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$$YD = \psi \tag{2.7}$$

where

and
 $D =$

$$\begin{pmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} & x_{n+2}^{11} \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 & 9x_{n+3}^8 & 10x_{n+3}^9 & 11x_{n+3}^{10} \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 & 72x_{n+3}^7 & 90x_{n+3}^8 & 110x_{n+3}^9 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & 720x_{n+3}^7 & 990x_{n+3}^8 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^2 & 840x_{n+3}^3 & 1680x_{n+3}^4 & 3024x_{n+3}^5 & 5040x_{n+3}^6 & 7920x_{n+3}^7 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 2520x_n^2 & 6720x_n^3 & 15120x_n^4 & 30240x_n^5 & 55440x_n^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{1}{8}} & 2520x_{n+\frac{1}{8}}^2 & 6720x_{n+\frac{1}{8}}^3 & 15120x_{n+\frac{1}{8}}^4 & 30240x_{n+\frac{1}{8}}^5 & 55440x_{n+\frac{1}{8}}^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & 2520x_{n+1}^2 & 6720x_{n+1}^3 & 15120x_{n+1}^4 & 30240x_{n+1}^5 & 55440x_{n+1}^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{9}{8}} & 2520x_{n+\frac{9}{8}}^2 & 6720x_{n+\frac{9}{8}}^3 & 15120x_{n+\frac{9}{8}}^4 & 30240x_{n+\frac{9}{8}}^5 & 55440x_{n+\frac{9}{8}}^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+2} & 2520x_{n+2}^2 & 6720x_{n+2}^3 & 15120x_{n+2}^4 & 30240x_{n+2}^5 & 55440x_{n+2}^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{17}{8}} & 2520x_{n+\frac{17}{8}}^2 & 6720x_{n+\frac{17}{8}}^3 & 15120x_{n+\frac{17}{8}}^4 & 30240x_{n+\frac{17}{8}}^5 & 55440x_{n+\frac{17}{8}}^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+3} & 2520x_{n+3}^2 & 6720x_{n+3}^3 & 15120x_{n+3}^4 & 30240x_{n+3}^5 & 55440x_{n+3}^6 \end{pmatrix}$$

Equation (2.7) is solved by matrix inversion technique to obtain a_j 's which are then substituted into (2.1) to get a continuous linear multistep method (LMM) of the form:

$$y(x) = \alpha_2(x)y_{n+2} + h\gamma_3(x)y'_{n+2} + h^2\delta_3(x)y''_{n+3} + h^3\sigma_3(x)y'''_{n+3} + h^4\mu_3(x)y^{iv}_{n+3} + h^5 \left(\beta_0(x)f_n + \beta_{\frac{1}{8}}(x)f_{n+\frac{1}{8}} + \beta_1(x)f_{n+1} + \beta_{\frac{9}{8}}(x)f_{n+\frac{9}{8}} + \beta_2(x)f_{n+2} + \beta_{\frac{17}{8}}(x)f_{n+\frac{17}{8}} + \beta_3(x)f_{n+3} \right) \tag{2.8}$$

where

$$\alpha_2(x) = 1 \tag{2.9}$$

$$\gamma_3(x) = -2h + x \tag{2.10}$$

$$\delta_3(x) = 4h^2 - 3hx + \frac{1}{2}x^2 \tag{2.11}$$

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$$\sigma_3(x) = -\frac{13}{3}h^3 + \frac{9}{2}h^2x + \frac{1}{2}x^3 - \frac{3}{2}hx^2 \quad (2.12)$$

$$\mu_3(x) = -\frac{9}{2}h^3x - \frac{1}{2}hx^3 + \frac{10}{3}h^4 + \frac{1}{24}x^4 + \frac{9}{4}h^2x^2 \quad (2.13)$$

$$\beta_0(x) = \left. \begin{aligned} &\frac{1}{120}x^5 - \frac{685}{44064}\frac{x^6}{h} + \frac{8289}{19040}h^3x^2 + \frac{13519}{1156680}\frac{x^7}{h^2} - \frac{3579}{4760}h^4x - \frac{685}{137088}\frac{x^8}{h^3} + \frac{6053}{342720}hx^4 \\ &+ \frac{2179}{1735020}\frac{x^9}{h^4} - \frac{355}{2856}h^2x^3 - \frac{5}{28917}\frac{x^{10}}{h^5} + \frac{19048537}{38170440}h^5 + \frac{16}{1590435}\frac{x^{11}}{h^6} \end{aligned} \right\} (2.14)$$

$$\beta_{\frac{1}{2}}(x) = \left. \begin{aligned} &-\frac{19764}{28175}h^3x^2 + \frac{1188}{5635}h^2x^3 - \frac{246}{5635}hx^4 + \frac{68}{4025}\frac{x^6}{h} + \frac{359}{50715}\frac{x^8}{h^3} + \frac{33858}{28175}h^4x \\ &-\frac{128}{8367975}\frac{x^{11}}{h^6} - \frac{19986364}{25103925}h^5 - \frac{842}{456435}\frac{x^9}{h^4} - \frac{3908}{253575}\frac{x^7}{h^2} + \frac{592}{2282175}\frac{x^{10}}{h^5} \end{aligned} \right\} (2.15)$$

$$\beta_1(x) = \left. \begin{aligned} &\frac{2403}{490}h^3x^2 + \frac{3119}{105840}\frac{x^7}{h^2} - \frac{4527}{3920}h^2x^3 + \frac{977}{15680}hx^4 - \frac{17681}{846720}\frac{x^8}{h^3} + \frac{2537273}{436590}h^5 \\ &+ \frac{16}{218295}\frac{x^{11}}{h^6} - \frac{3429}{392}h^4x + \frac{1643}{238140}\frac{x^9}{h^4} - \frac{17}{1680}\frac{x^6}{h} - \frac{67}{59535}\frac{x^{10}}{h^5} \end{aligned} \right\} (2.16)$$

$$\beta_{\frac{9}{8}}(x) = \left. \begin{aligned} &-\frac{8408}{297675}\frac{x^7}{h^2} + \frac{136}{14175}\frac{x^6}{h} - \frac{66400184}{9823275}h^5 - \frac{7272}{1225}h^3x^2 + \frac{352}{297675}\frac{x^{10}}{h^5} + \frac{1112}{735}h^2x^3 \\ &-\frac{256}{3274425}\frac{x^{11}}{h^6} - \frac{268}{2205}hx^4 + \frac{46}{2205}\frac{x^8}{h^3} - \frac{1268}{178605}\frac{x^9}{h^4} + \frac{12612}{1225}h^4x \end{aligned} \right\} (2.17)$$

$$\beta_2(x) = \left. \begin{aligned} &-\frac{58239}{4900}h^4x + \frac{247}{79380}\frac{x^9}{h^4} - \frac{17}{5600}\frac{x^6}{h} + \frac{381}{3136}hx^4 - \frac{59}{99225}\frac{x^{10}}{h^5} + \frac{409}{44100}\frac{x^7}{h^2} \\ &+ \frac{252801}{39200}h^3x^2 + \frac{70672219}{8731800}h^5 - \frac{2213}{282240}\frac{x^8}{h^3} - \frac{297}{196}h^2x^3 + \frac{16}{363825}\frac{x^{11}}{h^6} \end{aligned} \right\} (2.18)$$

$$\beta_{\frac{17}{8}}(x) = \left. \begin{aligned} &-\frac{634}{1465}hx^4 - \frac{6159940}{742203}h^5 - \frac{128}{3711015}\frac{x^{11}}{h^6} + \frac{464}{1012095}\frac{x^{10}}{h^5} + \frac{659}{112455}\frac{x^8}{h^3} + \frac{7164}{4165}h^2x^3 \\ &-\frac{772}{112455}\frac{x^7}{h^2} + \frac{51894}{4165}h^4x + \frac{4}{1785}\frac{x^6}{h} - \frac{29052}{4165}h^3x^2 - \frac{2386}{1012095}\frac{x^9}{h^4} \end{aligned} \right\} (2.19)$$

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$$\beta_3(x) = \left. \begin{aligned} &-\frac{1957}{2163}hx^4 - \frac{95877}{225400}h^2x^2 + \frac{1815}{18032}h^2x^3 + \frac{181017}{225400}x^4x - \frac{85219667}{150623550}h^5 - \frac{527}{6491520}x^8 \\ &-\frac{17}{579600}x^6 + \frac{191}{5477220}x^9 - \frac{17}{2282175}x^{10} + \frac{3323}{36514800}x^7 + \frac{16}{25103925}x^{11} \end{aligned} \right\} \quad (2.20)$$

Equations (2.9) to (2.20) are then substituted into (2.8) to get the required continuous scheme and then evaluating same, that is (2.8) at $x = x_{n+j}, j = 0, \frac{1}{8}, 1, \frac{9}{8}, \frac{17}{8}, 3$ gives the 3-step block method with 3 off-grid points as follows:

$$\begin{aligned} y_n = & y_{n+2} - 2hz_{n+3} + 4h^2g_{n+3} - \frac{13}{3}h^3p_{n+3} + \frac{10}{3}h^4q_{n+3} + \frac{19048537}{38170440}h^5f_n - \frac{19986364}{25103925}h^5f_{n+\frac{1}{8}} \\ & + \frac{2537273}{436590}h^5f_{n+1} - \frac{66400184}{9823275}h^5f_{n+\frac{9}{8}} + \frac{70672219}{871800}h^5f_{n+2} - \frac{6159940}{742203}h^5f_{n+\frac{17}{8}} \\ & - \frac{85219667}{150623550}h^5f_{n+3} \end{aligned} \quad (2.21)$$

$$\begin{aligned} y_{n+\frac{1}{8}} = & y_{n+2} - \frac{15}{8}hz_{n+3} + \frac{465}{128}h^2g_{n+3} - \frac{3885}{1024}h^3p_{n+3} + \frac{91915}{32768}h^4q_{n+3} \\ & + \frac{13389039703513}{32527934816256}h^5f_n - \frac{73732555015061}{112313059246080}h^5f_{n+\frac{1}{8}} + \frac{449332616642293}{93756988588032}h^5f_{n+1} \\ & - \frac{40743024285259}{7324764733440}h^5f_{n+\frac{9}{8}} + \frac{1047860425868189}{156261647646720}h^5f_{n+2} - \frac{341075901208237}{49808400187392}h^5f_{n+\frac{17}{8}} \\ & - \frac{1695817796714089}{3594017895874560}h^5f_{n+3} \end{aligned} \quad (2.22)$$

$$\begin{aligned} y_{n+1} = & y_{n+2} - hz_{n+3} + \frac{3}{2}h^2g_{n+3} - \frac{7}{6}h^3p_{n+3} + \frac{5}{8}h^4q_{n+3} + \frac{6668141}{87246720}h^5f_n \\ & - \frac{30577253}{25103925}h^5f_{n+\frac{1}{8}} + \frac{73764487}{83825280}h^5f_{n+1} - \frac{9947962}{9823275}h^5f_{n+\frac{9}{8}} + \frac{176633803}{139708800}h^5f_{n+2} \\ & - \frac{13895509}{11133045}h^5f_{n+\frac{17}{8}} - \frac{929594089}{9639907200}h^5f_{n+3} \end{aligned} \quad (2.23)$$

$$\begin{aligned}
 y_{n+\frac{9}{8}} = & y_{n+2} - \frac{7}{8}hz_{n+3} + \frac{161}{128}h^2g_{n+3} - \frac{2863}{3072}h^3p_{n+3} + \frac{46529}{98304}h^4q_{n+3} \\
 & + \frac{81815448864709}{1463757066731520}h^5f_n - \frac{21432801934717}{240670841241600}h^5f_{n+\frac{1}{8}} + \frac{43032185697167}{66969277562880}h^5f_{n+1} \\
 & - \frac{6960624359189}{9417554657280}h^5f_{n+\frac{9}{8}} + \frac{309279545969173}{334846387814400}h^5f_{n+2} - \frac{32265908186951}{35577428705280}h^5f_{n+\frac{17}{8}} \\
 & + \frac{331803111681127}{4620880151838720}h^5f_{n+3}
 \end{aligned} \tag{2.24}$$

$$\begin{aligned}
 y_{n+\frac{17}{8}} = & y_{n+2} + \frac{1}{8}hz_{n+3} - \frac{15}{128}h^2g_{n+3} + \frac{169}{3072}h^3p_{n+3} - \frac{565}{32768}h^4q_{n+3} \\
 & - \frac{13996536033853}{10246299467120640}h^5f_n + \frac{3663477246437}{1684695888691200}h^5f_{n+\frac{1}{8}} - \frac{21845333749733}{1406354828820480}h^5f_{n+1} \\
 & + \frac{5874086157937}{329614413004800}h^5f_{n+\frac{9}{8}} - \frac{50420771345933}{2343924714700800}h^5f_{n+2} + \frac{15177373591421}{747126002810880}h^5f_{n+\frac{17}{8}} \\
 & + \frac{3492285491191627}{161730805314355200}h^5f_{n+3}
 \end{aligned} \tag{2.25}$$

$$\begin{aligned}
 y_{n+3} = & y_{n+2} + hz_{n+3} - \frac{1}{2}h^2g_{n+3} + \frac{1}{6}h^3p_{n+3} - \frac{1}{24}h^4q_{n+3} - \frac{1681157}{610727040}h^5f_n + \frac{109979}{25103925}h^5f_{n+\frac{1}{8}} \\
 & - \frac{871891}{27941760}h^5f_{n+1} + \frac{70286}{196455}h^5f_{n+\frac{9}{8}} - \frac{5957429}{139708800}h^5f_{n+2} + \frac{148537}{3711015}h^5f_{n+\frac{17}{8}} \\
 & + \frac{9156971}{1927981440}h^5f_{n+3}
 \end{aligned} \tag{2.26}$$

where $z_{n+3} = y'_{n+3}$, $g_{n+3} = y''_{n+3}$, $p_{n+3} = y'''_{n+3}$, $q_{n+3} = y^{iv}_{n+3}$.

Finding the first derivative of (2.8) and evaluating at gives the following discrete schemes:

$$\begin{aligned}
 z_n = & z_{n+3} - 3hg_{n+3} + \frac{9}{2}h^2p_{n+3} - \frac{9}{3}h^3q_{n+2} - \frac{3579}{4760}h^4f_n + \frac{33858}{28175}h^4f_{n+\frac{1}{8}} - \frac{3429}{392}h^4f_{n+1} \\
 & + \frac{12612}{1225}h^4f_{n+\frac{9}{8}} - \frac{58239}{4900}h^4f_{n+2} + \frac{51894}{4165}h^4f_{n+\frac{17}{8}} + \frac{181017}{225400}h^4f_{n+3}
 \end{aligned} \tag{2.27}$$

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$$\left. \begin{aligned}
 z_{n+\frac{1}{8}} &= z_{n+3} - \frac{23}{8}hg_{n+3} + \frac{529}{128}h^2p_{n+3} - \frac{12167}{3072}h^3q_{n+3} - \frac{6294634036169}{9702935101440}h^4f_n \\
 &+ \frac{431112130801}{416179814400}h^4f_{n+\frac{1}{8}} - \frac{30264901254827}{3995326218240}h^4f_{n+1} + \frac{5544388150717}{624269721600}h^4f_{n+\frac{9}{8}} \\
 &- \frac{68875029680251}{66588770330400}h^4f_{n+2} + \frac{45825841762951}{4245034106880}h^4f_{n+\frac{17}{8}} + \frac{4670528660017}{6658877030400}h^4f_{n+3}
 \end{aligned} \right\} (2.28)$$

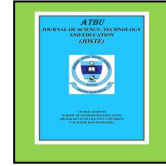
$$\left. \begin{aligned}
 z_{n+1} &= z_{n+3} - 2hg_{n+3} + 2h^2p_{n+3} - \frac{4}{3}h^3q_{n+3} - \frac{26536}{144585}h^4f_n + \frac{667648}{2282175}h^4f_{n+\frac{1}{8}} - \frac{126478}{59535}h^4f_{n+1} \\
 &+ \frac{729088}{297675}h^4f_{n+\frac{9}{8}} - \frac{301736}{99225}h^4f_{n+2} + \frac{618496}{202419}h^4f_{n+\frac{17}{8}} + \frac{497992}{2282175}h^4f_{n+3}
 \end{aligned} \right\} (2.29)$$

$$\left. \begin{aligned}
 z_{n+\frac{9}{8}} &= z_{n+3} - \frac{15}{8}hg_{n+3} + \frac{225}{128}h^2p_{n+3} - \frac{1125}{1024}h^3q_{n+3} - \frac{166565625}{1140850688}h^4f_n + \frac{1099929375}{4726980608}h^4f_{n+\frac{1}{8}} \\
 &- \frac{5544028125}{3288334336}h^4f_{n+1} + \frac{199548375}{102760448}h^4f_{n+\frac{9}{8}} - \frac{7973386875}{3288334336}h^4f_{n+2} + \frac{8456315625}{3493855232}h^4f_{n+\frac{17}{8}} \\
 &+ \frac{13371150375}{75631689728}h^4f_{n+3}
 \end{aligned} \right\} (2.30)$$

$$\left. \begin{aligned}
 z_{n+2} &= z_{n+3} - hg_{n+3} + \frac{1}{2}h^2p_{n+3} - \frac{1}{6}h^3q_{n+3} - \frac{1157}{182620}h^4f_n + \frac{50882}{2282175}h^4f_{n+\frac{1}{8}} - \frac{75949}{476280}h^4f_{n+1} \\
 &+ \frac{54476}{297675}h^4f_{n+\frac{9}{8}} - \frac{176417}{793800}h^4f_{n+2} + \frac{213154}{1012095}h^4f_{n+\frac{17}{8}} + \frac{391987}{18257400}h^4f_{n+3}
 \end{aligned} \right\} (2.31)$$

$$\left. \begin{aligned}
 z_{n+\frac{17}{8}} &= z_{n+3} - \frac{7}{8}hg_{n+3} + \frac{49}{128}h^2p_{n+3} - \frac{343}{3072}h^3q_{n+3} - \frac{11382658399}{1386133585920}h^4f_n \\
 &+ \frac{2553336247}{195349708800}h^4f_{n+\frac{1}{8}} - \frac{7600478347}{81537269760}h^4f_{n+1} + \frac{272391049}{2548039680}h^4f_{n+\frac{9}{8}} \\
 &- \frac{17402589659}{135895449600}h^4f_{n+2} + \frac{10433456663}{86633349120}h^4f_{n+\frac{17}{8}} + \frac{8441827163}{625119068160}h^4f_{n+3}
 \end{aligned} \right\} (2.32)$$

Finding the second derivative of (2.8) and evaluating at gives the following discrete schemes:



$$\left. \begin{aligned} g_n &= g_{n+3} - 3hp_{n+3} + \frac{9}{2}h^2q_{n+3} + \frac{8289}{9520}h^3f_n - \frac{39528}{28175}h^3f_{n+\frac{1}{8}} - \frac{2403}{245}h^3f_{n+1} \\ &\quad - \frac{14544}{1225}h^3f_{n+\frac{9}{8}} + \frac{252801}{19600}h^3f_{n+2} - \frac{58104}{4165}h^3f_{n+\frac{17}{8}} - \frac{95877}{112700}h^3f_{n+3} \end{aligned} \right\} \quad (2.33)$$

$$\left. \begin{aligned} g_{n+\frac{1}{8}} &= g_{n+3} - \frac{23}{8}hp_{n+3} + \frac{529}{128}h^2q_{n+3} + \frac{947255908339}{1212866887680}h^3f_n \\ &\quad - \frac{1551888683}{1238630400}h^3f_{n+\frac{1}{8}} + \frac{212933535469}{23781703680}h^3f_{n+1} - \frac{419849090233}{39016857600}h^3f_{n+\frac{9}{8}} \\ &\quad + \frac{3269591444227}{277453209600}h^3f_{n+2} - \frac{1122195711079}{88438210560}h^3f_{n+\frac{17}{8}} - \frac{646687401811}{832359628800}h^3f_{n+3} \end{aligned} \right\} \quad (2.34)$$

$$\left. \begin{aligned} g_{n+1} &= g_{n+3} - 2hp_{n+3} + 2h^2q_{n+3} + \frac{9374}{28917}h^3f_n - \frac{56192}{108675}h^3f_{n+\frac{1}{8}} + \frac{1537}{405}h^3f_{n+1} \\ &\quad - \frac{1308928}{297675}h^3f_{n+\frac{9}{8}} + \frac{175258}{33075}h^3f_{n+2} - \frac{1849472}{337365}h^3f_{n+\frac{17}{8}} - \frac{810227}{2282175}h^3f_{n+3} \end{aligned} \right\} \quad (2.35)$$

$$\left. \begin{aligned} g_{n+\frac{9}{8}} &= g_{n+3} - \frac{15}{8}hp_{n+3} + \frac{225}{128}h^2q_{n+3} + \frac{276508125}{998244352}h^3f_n - \frac{130501125}{295436288}h^3f_{n+\frac{1}{8}} \\ &\quad + \frac{1327269375}{411041792}h^3f_{n+1} - \frac{23970375}{6422528}h^3f_{n+\frac{9}{8}} + \frac{1877792625}{411041792}h^3f_{n+2} \\ &\quad - \frac{1024329375}{218365952}h^3f_{n+\frac{17}{8}} - \frac{2913688125}{9453961216}h^3f_{n+3} \end{aligned} \right\} \quad (2.36)$$

$$\left. \begin{aligned} g_{n+2} &= g_{n+3} - hp_{n+3} + \frac{1}{2}h^2q_{n+3} + \frac{7519}{136080}h^3f_n - \frac{66952}{760725}h^3f_{n+\frac{1}{8}} + \frac{25117}{39690}h^3f_{n+1} \\ &\quad - \frac{216464}{297675}h^3f_{n+\frac{9}{8}} + \frac{479279}{529200}h^3f_{n+2} - \frac{17288}{19845}h^3f_{n+\frac{17}{8}} - \frac{676189}{9128700}h^3f_{n+3} \end{aligned} \right\} \quad (2.37)$$



$$\left. \begin{aligned}
 g_{n+\frac{17}{8}} &= g_{n+3} - \frac{7}{8} h p_{n+3} + \frac{49}{128} h^2 q_{n+3} + \frac{1316710801}{34653339648} h^3 f_n \\
 &- \frac{246253763}{4069785600} h^3 f_{n+\frac{1}{8}} + \frac{1474048331}{3397386240} h^3 f_{n+1} - \frac{396724433}{796262400} h^3 f_{n+\frac{9}{8}} \\
 &+ \frac{3464323667}{5662310400} h^3 f_{n+2} - \frac{1051652063}{1804861440} h^3 f_{n+\frac{17}{8}} - \frac{21070617253}{390699417600} h^3 f_{n+3}
 \end{aligned} \right\} (2.38)$$

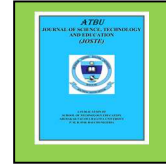
Finding the third derivative of (2.8) and evaluating at gives the following discrete schemes:

$$\left. \begin{aligned}
 p_n &= p_{n+3} - 3h q_{n+3} - \frac{355}{476} h^2 f_n + \frac{128}{5635} h^2 f_{n+\frac{1}{8}} - \frac{13581}{1960} h^2 f_{n+1} + \frac{2224}{245} h^2 f_{n+\frac{9}{8}} \\
 &- \frac{891}{98} h^2 f_{n+2} + \frac{42984}{4165} h^2 f_{n+\frac{17}{8}} + \frac{5445}{9016} h^2 f_{n+3}
 \end{aligned} \right\} (2.39)$$

$$\left. \begin{aligned}
 p_{n+\frac{1}{8}} &= p_{n+3} - \frac{23}{8} h q_{n+3} - \frac{17386801171}{25268060160} h^2 f_n \\
 &+ \frac{32867299}{28901376} h^2 f_{n+\frac{1}{8}} - \frac{7795530737}{1156055040} h^2 f_{n+1} + \frac{5666500409}{6550280960} h^2 f_{n+\frac{9}{8}} \\
 &- \frac{2017933451}{231211008} h^2 f_{n+2} + \frac{24231712031}{2456616960} h^2 f_{n+\frac{17}{8}} + \frac{2000406623}{3468165120} h^2 f_{n+3}
 \end{aligned} \right\} (2.40)$$

$$\left. \begin{aligned}
 p_{n+1} &= p_{n+3} - 2h q_{n+3} - \frac{18962}{48195} h^2 f_n + \frac{10624}{16905} h^2 f_{n+\frac{1}{8}} - \frac{6266}{1323} h^2 f_{n+1} \\
 &+ \frac{110848}{19845} h^3 f_{n+\frac{9}{8}} - \frac{4442}{735} h^2 f_{n+2} + \frac{738944}{112455} h^2 f_{n+\frac{17}{8}} + \frac{59036}{152145} h^2 f_{n+3}
 \end{aligned} \right\} (2.41)$$

$$\left. \begin{aligned}
 p_{n+\frac{9}{8}} &= p_{n+3} - \frac{15}{8} h q_{n+3} - \frac{3220625}{8912896} h^2 f_n + \frac{1852875}{3211264} h^2 f_{n+\frac{1}{8}} - \frac{110953125}{25690112} h^2 f_{n+1} \\
 &+ \frac{8115925}{1605632} h^2 f_{n+\frac{9}{8}} - \frac{144979875}{25690112} h^2 f_{n+2} + \frac{332386875}{54591488} h^2 f_{n+\frac{17}{8}} \\
 &+ \frac{9279225}{25690112} h^2 f_{n+3}
 \end{aligned} \right\} (2.42)$$



$$\left. \begin{aligned}
 p_{n+2} = p_{n+3} - hq_{n+3} - \frac{1046}{6885}h^2f_n + \frac{2456}{10143}h^2f_{n+\frac{1}{8}} - \frac{10309}{5880}h^2f_{n+1} + \frac{40048}{19845}h^2f_{n+\frac{9}{8}} \\
 - \frac{4591}{1764}h^2f_{n+2} + \frac{32152}{12495}h^2f_{n+\frac{17}{8}} + \frac{212269}{2117160}h^2f_{n+3}
 \end{aligned} \right\} \quad (2.43)$$

$$\left. \begin{aligned}
 p_{n+\frac{17}{8}} = p_{n+3} - \frac{7}{8}hq_{n+3} - \frac{448288309}{3609722880}h^2f_n + \frac{894887}{4521984}h^2f_{n+\frac{1}{8}} - \frac{101205923}{70778880}h^2f_{n+1} \\
 + \frac{21830921}{13271040}h^2f_{n+\frac{9}{8}} - \frac{16495213}{7864320}h^2f_{n+2} + \frac{61473097}{30081024}h^2f_{n+\frac{17}{8}} + \frac{240735481}{1627914240}h^2f_{n+3}
 \end{aligned} \right\} \quad (2.44)$$

Finding the fourth derivative of (2.8) and evaluating at gives the following discrete schemes:

$$\left. \begin{aligned}
 q_n = q_{n+3} + \frac{6053}{14280}hf_n - \frac{5904}{5635}h^2f_{n+\frac{1}{8}} + \frac{2931}{1960}hf_{n+1} - \frac{2144}{735}hf_{n+\frac{9}{8}} + \frac{1143}{392}hf_{n+2} \\
 - \frac{15216}{4165}hf_{n+\frac{17}{8}} - \frac{1957}{9016}hf_{n+3}
 \end{aligned} \right\} \quad (2.45)$$

$$\left. \begin{aligned}
 q_{n+\frac{1}{8}} = q_{n+3} - \frac{755947877}{1579253760}hf_n - \frac{2202043}{2257920}hf_{n+\frac{1}{8}} + \frac{320563949}{21676030}hf_{n+1} \\
 - \frac{29480641}{10160640}hf_{n+\frac{9}{8}} + \frac{210330929}{72253440}hf_{n+2} - \frac{420284681}{115153920}hf_{n+\frac{17}{8}} - \frac{47059633}{216760320}hf_{n+3}
 \end{aligned} \right\} \quad (2.46)$$

$$\left. \begin{aligned}
 q_{n+1} = q_{n+3} + \frac{12368}{48195}hf_n - \frac{20864}{50715}hf_{n+\frac{1}{8}} + \frac{21811}{6615}hf_{n+1} - \frac{84736}{19845}hf_{n+\frac{9}{8}} \\
 + \frac{1412}{441}hf_{n+2} - \frac{434048}{112455}hf_{n+\frac{17}{8}} - \frac{32651}{152145}hf_{n+3}
 \end{aligned} \right\} \quad (2.47)$$

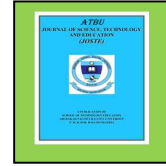
$$\left. \begin{aligned}
 q_{n+\frac{9}{8}} = q_{n+3} + \frac{3005875}{11688176}hf_n - \frac{475425}{1154048}hf_{n+\frac{1}{8}} + \frac{5392875}{1605632}hf_{n+1} - \frac{316535}{75264}hf_{n+\frac{9}{8}} \\
 + \frac{5138325}{16056322}hf_{n+2} - \frac{3291375}{852992}hf_{n+\frac{17}{8}} - \frac{7925665}{36929536}hf_{n+3}
 \end{aligned} \right\} \quad (2.48)$$

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$$q_{n+2} = q_{n+3} + \frac{85487}{385560} h^2 f_n - \frac{17936}{50715} h f_{n+\frac{1}{8}} + \frac{136769}{52920} h f_{n+1} - \frac{59168}{19845} h f_{n+\frac{9}{8}} + \frac{70499}{17640} h f_{n+2} - \frac{478544}{112455} h f_{n+\frac{17}{8}} - \frac{258179}{1217160} h f_{n+3} \quad (2.49)$$

$$q_{n+\frac{17}{8}} = q_{n+3} + \frac{50096179}{225607680} h f_n - \frac{375389}{1059840} h f_{n+\frac{1}{8}} + \frac{11453309}{44233680} h f_{n+1} - \frac{619409}{207360} h f_{n+\frac{9}{8}} + \frac{5993729}{1474560} h f_{n+2} - \frac{9864281}{2350080} h f_{n+\frac{17}{8}} - \frac{21586327}{101744640} h f_{n+3} \quad (2.50)$$

Analysis of the Basic Properties of the block Method

The properties of the block method are analyzed as follows:

Order and Error Constants of the block method

The technique of Lambert (1973) and Fatunla (1991) for determining the order of a numerical discrete schemes is used to find the order of the new block method. Therefore, the new block method is of equal order, that is $p = [7, 7, 7, 7, 7, 7]^T$ with error constants

$$\left[\begin{array}{c} \frac{102937753}{214592716800}, \frac{6090747615530471}{15361145010263162880}, \frac{21330493}{286123622400}, \frac{71912769732295}{1316669572308271104}, \\ \frac{104227719982973}{76805725051315814400}, \frac{94385}{34334834688} \end{array} \right]^T$$

Zero stability of the block method

Definition: A block method is said to be zero-stable if the roots of the first characteristic polynomial and satisfies where

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \text{ and } A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ are the coefficients of}$$

$$\left[y_{n+3}, y_{n+\frac{17}{8}}, y_{n+\frac{9}{8}}, y_{n+\frac{1}{8}}, y_{n+2}, y_{n+1} \right]^T \text{ and } \left[y_{n-3}, y_{n-\frac{17}{8}}, y_{n-\frac{9}{8}}, y_{n-\frac{1}{8}}, y_{-n}, y_{n-1} \right]^T \text{ respectively}$$



Normalizing and we have $A^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $A^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

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Hence, the block method is zero stable since it satisfies

Consistency

Definition: A linear multistep method is said to be consistent if the following conditions are satisfied:

i) The order p of accuracy is greater than one (

ii) $\sum_{j=0}^k \alpha_j = 0$ (Lambert, 1973)

It is clear that the block method is consistent since and $\alpha_0 + \alpha_{\frac{1}{8}} + \alpha_1 + \alpha_{\frac{9}{8}} + \alpha_2 + \alpha_{\frac{17}{8}} + \alpha_3 = 0$

Convergence

The necessary and sufficient condition for linear multistep method to be convergent is for it to be consistent and zero stable. Following this theorem, the block method developed is convergent.

Numerical Examples

In this section, some numerical examples are given to illustrate the efficiency of the proposed method. We find the absolute errors of the approximate solution and also make comparisons with some existing methods in the literature.

Problem 1 (SIR model): the SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people S(t), no of people infected I (t) and the number of people who have recovered R (t). This is a good and simple model for many infectious diseases

including measles, mumps and rubella. It is given by the following three coupled equations:

Where are positive parameters? Define y to be:
. Adding (1) - (3), the following evolution equation for y is obtained.

Taking and attaching an initial condition (for a particular closed population), the equation below is achieved.

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Exact Solution

Source: John et al (2022).

Problem 2: Consider the stiff problem

Exact solution: Source: Raymond (2020)

Problem 3: Nonlinear third order ODE problem:

Exact Solution Source: Oluwaseun and Zurni (2019)

Problem 4: Nonlinear 4th order ODE problem:

Exact solution:

Source: Victor (2021) and Olusola (2022).

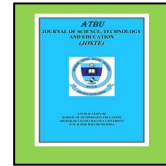
Problem 5: Consider linear fifth order problem

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Exact solution: Source: Roseline et al (2021).

Table 1: Comparing the exact Solution with the Proposed Methods for Problem 1

x	Exact Solution	Computed Solution	Absolute Error
0.1	0.5243852877496429954542874	0.5243852877496429954110886	4.3199
0.2	0.5475812909820202134178754	0.5475812909820202133687724	4.9103
0.3	0.5696460117874710963854831	0.5696460117874710963384788	4.7004
0.4	0.5906346234610090706650322	0.5906346234610090705831393	8.1893
0.5	0.6105996084642975658774148	0.6105996084642975657926203	8.4795
0.6	0.6295908896591410669665631	0.6295908896591410668856492	8.0914
0.7	0.6476559551406432828225896	0.6476559551406432827136200	1.0897
0.8	0.6648399769821803496277836	0.6648399769821803495181934	1.096
0.9	0.6811859241891133534281283	0.6811859241891133533236636	1.0446
1.0	0.6967346701436832881981002	0.6967346701436832880711864	1.2691

Table 2: Comparing the exact Solution with the Proposed Methods for Problem 2

X	Exact Solution	Computed Solution	Absolute Error
0.1	0.9048374180359595731642491	0.9048374180359595731038462	6.0403
0.2	0.8187307530779818586699355	0.81873075307798185860232650	6.7609
0.3	0.7408182206817178660668738	0.74081822068171786600161744	6.5256
0.4	0.6703200460356393007444329	0.67032004603563930063919360	1.0524
0.5	0.6065306597126334236037995	0.60653065971263342349881706	1.0498
0.6	0.5488116360940264326284589	0.54881163609402643253040799	9.8051
0.7	0.4965853037914095147048001	0.49658530379140951458184843	1.2295
0.8	0.4493289641172215914301024	0.44932896411722159131162100	1.1848
0.9	0.4065696597405991118834542	0.40656965974059911177398152	1.0947
1.0	0.3678794411714423215955238	0.36787944117144232147110935	1.2441

Table 3: Comparing the exact Solution with the Proposed Method for Problem 3

	Exact Solution	Computed Solution	Absolute Error
0.1	0.09983341664682815230681419	0.09983341664682815230681419	1.50
0.2	0.19866933079506121545941263	0.19866933079506121545941264	2.05
0.3	0.29552020666133957510532075	0.29552020666133957510532085	9.93
0.4	0.38941834230865049166631175	0.38941834230865049166631207	3.08
0.5	0.47942553860420300027328793	0.47942553860420300027328868	7.47
0.6	0.56464247339503535720094544	0.56464247339503535720094699	1.55
0.7	0.64421768723769105367261435	0.64421768723769105367261723	2.88
0.8	0.71735609089952276162717461	0.71735609089952276162717955	4.94
0.9	0.78332690962748338846138232	0.78332690962748338846139029	7.97
1.0	0.84147098480789650665250232	0.84147098480789650665251459	1.23

Table 4: Comparing the exact Solution with the Proposed Methods for Problem 4

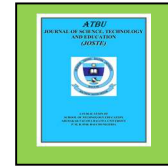
	Exact Solution	Computed Solution	Absolute Error
0.1	1.11517091807564762481170782649	1.11517091807564762481170773505	9.144
0.2	1.26140275816016983392107199464	1.26140275816016983392107058431	1.410
0.3	1.43985880757600310398374431333	1.43985880757600310398373727969	7.034
0.4	1.65182469764127031782485295284	1.65182469764127031782483093908	2.201
0.5	1.89872127070012814684865078781	1.89872127070012814684865078781	5.331

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0.6	2.18211880039050897487536766816	2.18211880039050897487525798237	1.096
0.7	2.50375270747047652162454938858	2.50375270747047652162434775980	2.016
0.8	2.86554092849246760457953753140	2.86554092849246760457919629631	3.412
0.9	3.26960311115694966380012656360	3.26960311115694966379958441159	5.422
1.0	3.71828182845904523536028747135	3.71828182845904523535946791934	8.196

Table 5: Comparing the exact Solution with the Proposed Methods for Problem 5

	Exact Solution	Computed Solution	Absolute Error
0.1	1.3212361748069979862	1.3212361748069986412	6.550
0.2	1.6904940284363315333	1.6904940284363702995	3.8766
0.3	2.1176390070518485500	2.1176390070521591966	3.1065
0.4	2.6149592708011180963	2.6149592708025389504	1.4209
0.5	3.1977073670632482357	3.1977073670683796625	5.1314
0.6	3.8847593961315828467	3.8847593961465231402	1.4940
0.7	4.6994176540823656409	4.6994176541189166031	3.6551
0.8	5.6703885152946375653	5.6703885153738786182	7.9241
0.9	6.8329743740404294722	6.8329743741973538051	1.5692
1.0	8.2305270837385467338	8.2305270840275704210	2.8902

Table 6: Compares the error of the Proposed Method for Problem 1 with John et al (2022)

	Exact Solution	Error in New method K = 3, h = 0.1, p = 7	Error in John et al. (2022) k = 1/2, h = 0.1, p = 6
0.1	0.5243852877496429954542874	4.319	4.939
0.2	0.5475812909820202134178754	4.910	9.399
0.3	0.5696460117874710963854831	4.700	1.341
0.4	0.5906346234610090706650322	8.189	1.701
0.5	0.6105996084642975658774148	8.479	2.022
0.6	0.6295908896591410669665631	8.091	2.308
0.7	0.6476559551406432828225896	1.089	2.592
0.8	0.6648399769821803496277836	1.096
0.9	0.6811859241891133534281283	1.045
1.0	0.6967346701436832881981002	1.269

Table 7: Compares the error of the Proposed Method for Problem 2 with Raymond (2020)

	Exact Solution	Error in New method	Raymond (2020)
0.1	0.9048374180359595731642491	6.0403	1.2420
0.2	0.8187307530779818586699355	6.7609	4.5800
0.3	0.7408182206817178660668738	6.5256	1.5361
0.4	0.6703200460356393007444329	1.0524	1.4527
0.5	0.6065306597126334236037995	1.0498	1.3834
0.6	0.5488116360940264326284589	9.8051	1.3277
0.7	0.4965853037914095147048001	1.2295	1.2858
0.8	0.4493289641172215914301024	1.1848	1.2566
0.9	0.4065696597405991118834542	1.0947	1.2397
1.0	0.3678794411714423215955238	1.2441	1.2354

Table 8: Compares the error of the Proposed Method for Problem 3 with Oluwaseun (2019)

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	Exact Solution	Error in New method	Error in Oluwaseun (2019)
0.1	0.09983341664682815230681419	1.50	6.66
0.2	0.19866933079506121545941263	2.05	1.91
0.3	0.29552020666133957510532075	9.93	1.24
0.4	0.38941834230865049166631175	3.08	2.89
0.5	0.47942553860420300027328793	7.47	5.60
0.6	0.56464247339503535720094544	1.55	9.69
0.7	0.64421768723769105367261435	2.88	1.55
0.8	0.71735609089952276162717461	4.94	2.33
0.9	0.78332690962748338846138232	7.97	3.35
1.0	0.84147098480789650665250232	1.23	4.64

Table 9: Compares the error of the Proposed Method for Problem 4 with Victor (2021)

	Exact Solution	Error in New method	Error in Victor (2021)
		k = 3, h = 0.1, p = 7	k = 15, h = 0.003125, p = 16
0.1	1.11517091807564762481170782649	9.144	2.220
0.2	1.26140275816016983392107199464	1.410	0.000
0.3	1.43985880757600310398374431333	7.034	2.220
0.4	1.65182469764127031782485295284	2.201	0.000
0.5	1.89872127070012814684865078781	5.331	0.000
0.6	2.18211880039050897487536766816	1.096	2.220
0.7	2.50375270747047652162454938858	2.016	6.661
0.8	2.86554092849246760457953753140	3.412	1.332
0.9	3.26960311115694966380012656360	5.422	2.887
1.0	3.71828182845904523536028747135	8.196

DISCUSSION OF RESULTS

The results produced by the developed three-step block method for solving linear and nonlinear from first to fifth orders of ODEs are displayed in tables 1 to 5. The errors generated are determined as absolute difference between the exact and approximated or computed solution. Remarkably minimal errors in the new method affirm its effectiveness.

Table 6 compares the result of the new hybrid block method with those achieved by John (2022) in terms of error for solving problem1, emphasizing the superior performance of the new method. Similarly, table 7 suggests that the new method outperformed the results obtained by Ezekiel (2018) for problem 2. Furthermore, upon closer scrutiny of tables 8 and 9, it becomes apparent that the new method notably surpasses the performance of Oluseun (2019) and Victor (2021) in terms of error when addressing problems 3 and 4, respectively.

CONCLUSION

New hybrid block method designed with step number k = 3 have been developed, enabling the solution of both linear and nonlinear ODEs up to fifth order. The derivation of continuous and discrete schemes forming the block is accomplished using interpolation and collocation techniques, resulting in a uniform order 7 for k = 3.

Following an examination of the method's fundamental properties, it was established that it meets the criteria for consistency, zero stability, and convergence. The efficacy of the newly developed method was assessed by testing on a variety of linear and nonlinear IVPs of ODEs spanning first through fifth orders. The results, detailed in tables 6 and 9, demonstrated its superiority in terms of accuracy compared to existing methods with which it was compared. Its primary advantage lies in its ability to directly address IVPs of ODEs ranging from first to fifth order.

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In addition, it removes the need to create distinct numerical methods for five different orders of differential equations, thereby reducing the time required for developing these methods. The new method is recommended as an effective and efficient numerical method for the direct solution of first through fifth orders IVP of ODEs.

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