

Orthogonal Polynomials: Essential Tools and Applications in Modern Mathematics

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ABSTRACT

Orthogonal Polynomials are classes of polynomials $p_n(x)$ defined over a range $[a, b]$ that obey an orthogonality relation. Where $w(x)$ is a weight function and δ_{mn} is the Kronecker delta. If $c_n = 1$, the polynomials are orthogonal and orthonormal. They are also defined as polynomials whose inner product equals zero. The basic properties, recurrence relations, distributions of zeros and other characterizations of such polynomials are included. The very classical orthogonal polynomials and some basic concepts of orthogonal polynomials were investigated with major forms of orthogonality on Legendre, Bessel, and Chebyshev's polynomials. Gram Schmidt and Sturm-Liouville's approach was used to show orthogonality in the three poly-nomials reviewed and illustrative examples of each were stated. Finally, some applications in numerical integration; interpolation, linear spaces, vector spaces and unit circles were studied.

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INTRODUCTION

Orthogonal polynomials arise in many areas of Mathematics and have been the subject of interest by many mathematicians. In recent years this interest has often arisen from outside the orthogonal polynomials community where several applications are found to be useful in the solution of Sciences, Engineering, Mathematical and Physical problems.

Polynomials appear in a wide variety of areas of Mathematics and Science. For example, they are used to form polynomial equations, which encode a wide range of problems, from elementary word problems to complicated problems in the sciences; However they are used to define polynomial functions, which appear in settings ranging from basic Chemistry and Physics to Economics and Social Science, they are used in calculus and Numerical Analysis to approximate other functions.

Also in Advanced Mathematics, polynomials, are used to construct polynomial rings and algebraic varieties, central concepts in algebraic geometry and some other related

applications which will be revealed in the chapters of this book.

Orthogonal polynomials typically are used as basic functions in which to expand other more complicated functions. In contrast, polynomials orthogonal with respect to general, nonstandard, weight functions and measures have received much less attention in applications, in part because of the considerable difficulties attending their numerical generation. Orthogonal polynomials are classes of polynomials $\{P_n(x)\}$ defined over a range $[a, b]$ that obey an orthogonality relation.

$$\int_a^b w(x) p_m(x) p_n(x) dx = \delta_{mn} c_n$$

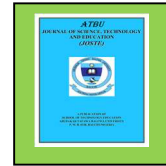
Where $w(x)$ is a weighting is the Kronecker delta. Just as Fourier series provide a convenient method of expanding a periodic function in a series of linearly independent terms, orthogonal polynomials provide a natural way to solve, expand and interpret solutions to many types of important differential equation. In mathematics, orthogonal polynomials sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to

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each other under some inner product i.e polynomial orthogonal with respect to an inner product.

The focus of this project will be from the orthogonal polynomials and revealing some of its applications in sciences; Mathematics and Physics in particular thereby introducing the most widely used orthogonal polynomials which definitely are the classical orthogonal polynomials, consisting of the Hermite Polynomials, the Jacobi Polynomials together with some of the general theory of orthogonal polynomials with the best known families and special cases of classical orthogonal polynomials which consists of Gegenbauer polynomials, the Chebyshev, and the Legendre polynomials. Some of the applications which I shall be considering in this project work includes the following;

Orthogonality in Geometry; it is aimed at finding two Euclidean vectors that are perpendicular to each other. Orthogonality on curves or functions in the plane; which is aimed at finding curves or functions that their tangent lines are perpendicular at that point.

Orthogonality in vector spaces; is aimed at getting two vectors X and Y in which their inner product is zero. And lastly, the connection between orthogonal polynomials and discrete integrable systems; The focus will be from the orthogonal polynomials point of view, looking at the kind of integrable systems that occur after studying the underlying structure of orthogonal polynomials.

BACKGROUND STUDY

The subject of orthogonal polynomials finds its origin in the 18th century, thanks to the works of Legendre, Laplace and Lagrange. While these three brilliant Mathematicians are remembered for their work in elliptic functions, the theory of differential equations and mathematical astronomy, they also developed the first examples of orthogonal polynomials, before any general theory existed. The development of the general theory of orthogonal polynomials began in the 19th century after investigations into *stieljes* continued fractions by *chebyshev*. Other important results found independent of the general theory were given by Gauss, Abel, Jacobi, Hermite and

Laguerre of whom the latter three gave their name to what became the *classical orthogonal polynomials*.

The classical orthogonal polynomials (referred to as the very classical polynomials in modern literature) were the first families of orthogonal polynomials to be established and are important because they were discovered to possess many more properties than other orthogonal polynomial systems of time. Orthogonal polynomials have since been found to have connections with trigonometric, hypergeometric, Bessel and elliptic functions; they have significance in helping to solve certain problems in quantum mechanics and Mathematical Statistics, and are related to important of interpolation and mechanical quadrature. However, there are some mathematicians who have worked on orthogonal polynomials, they are; Gabor Szego, Segei Bernstein, Naurn Akhiezer, Arthur Erdelyi, Yakov Geronimus, Wolfgang Hahn, Theodore Seio chimara, Mourad Ismial, Waleed Al-salam and Richard Askey.

The underlying orthogonal polynomials $\psi_n = P_n(\alpha, \alpha)$ (known, under different normalization, as Gegenbauer polynomials.) and their special cases include Legendre polynomials and Chebyshev polynomials of first and second kind. This approach rests upon three conceptual (and often counter-intuitive) steps, which will be presented in a general setting.

MOTIVATION

The study of orthogonal polynomials is one of the areas of Mathematics that has various applications especially in Sciences, Engineering, Mathematical and physical problems. To study the orthogonality of a polynomial, the basic knowledge of functions and the connections of orthogonal polynomials with other branches of mathematics need to be known. Therefore, the motivation of this study is the need to study the relationship between functions and some of the applications of polynomials particularly in Mathematics.

PROBLEM STATEMENT

The numerous uses and applications of orthogonal polynomials in Mathematics are

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evident in the avalanche of their types each of which has its peculiar characteristics but with general base structure of a polynomial. Orthogonal polynomials dwell between Pure and Applied Mathematics and embody fundamental theorem of calculus particularly the concept of integration of definite limits on which weight functions are placed. The existence of various functions which could be generated between one type and the other leave much to be desired in the study of orthogonal polynomials particularly when applying various fundamental concepts of generating functions differential equations, inner product and recursions.

Objectives

The study is aimed at achieving the following;

1. Reviewing some literature on orthogonal polynomials.
2. Exploring the basic concepts/properties of orthogonal polynomial.
3. Relating the connections that different special functions have with orthogonal polynomials.
4. Reviewing some applications of orthogonal polynomials.
5. Solving the orthogonality of different polynomials.

PRELIMINARIES AND DEFINITIONS

POLYNOMIAL

Definition 1.1 *In Mathematics, a polynomial is an expression consisting of variables, called indeterminate and coefficients that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents. An example of a polynomial of a single in- determinate(or variable) X, is $x^2 - 4x + 7$ which is quadratic polynomial.*

A polynomial in a single indeterminate can be written in the form.

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

where a_0, \dots, a_n are numbers. or more generally elements of a ring, and x is a symbol which is

called an indeterminate or, for historical reasons, a variable. The symbol x does not represent any value, although the usual(commutative, distributive) law valid for arithmetic operations also apply to it. This can be expressed more concisely by using summation notation:

$$\sum_{i=0}^n a_i x^i$$

That is, a polynomial can either be zero or can be written as the sum of a finite number of non-zero terms. Each term consists of the product of a number called the coefficient of the term and a finite number of indeterminate, raised to integer powers. The exponent on an indeterminate in a term is called the degree of that indeterminate in that term; the degree of the term is the sum of the degrees of the indeterminate in that term, and the degree of the polynomial is the largest degree of any one term with non-zero coefficient. Since $x = x^1$, the degree of an indeterminate without a written exponent is one. A term and a polynomial with no indeterminate are called respectively a constant polynomial, the degree of a constant term and of a non-zero constant polynomial is 0. The degree of the zero polynomial(which has no term) is not defined.

Example 1.1

$$-5x^2 y, 3xy^2$$

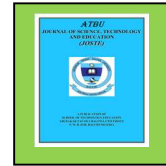
is a term. The coefficient are -5 and 3, the indeterminate are x and y , the degree of $x(-5x^2y)$ is two and the degree of $y(3xy^2)$ is two while the degree of the y and x in the above examples is one respectively. The degree of the entire term is the sum of the degrees of each indeterminate in it, so in this example the degree is $2 + 1 = 3$ and $1 + 2 = 3$. A monomial is a polynomial that consists of exactly one term. A binomial is a polynomial that consists of exactly two terms, while trinomial is a polynomial that consists of three terms. A polynomial P of exactly degree K is said to be monic if the coefficient of the monomial of highest degree is 1. Polynomials in the set are said to be monic orthogonal polynomials if they are

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orthogonal, monic and their norm are strictly positive.

The theory of inner product spaces enables an interpretation of orthogonal systems which play an important role in many branches of Mathematics, Physics and other applied and computational sciences. We start with the following definition:

Definition 1.2 Given a real linear space of functions X , an inner product (f, g) defined is a mapping of X^2 into \mathbb{R} such that:

- $(f + g, h) = (f, g) + (g, h)$ (linearity),
- $(\alpha f, g) = \alpha(f, g)$ (Homogeneity),
- $(f, g) = (g, f)$ (Symmetry),
- $(f, f) > 0, (f, f) = 0 \iff f = 0$ (Positive),

Where $f, g, h \in X$ and α is a real parameter. The space X will be called an inner product space.

A similar definition can be done for complex spaces. Namely, if X is a complex linear space, then the inner product $(f, g) : X^2 \rightarrow \mathbb{C}$ is such that the condition (c) is replaced by

$$(C) \quad (f, g) = (g, \bar{f}) \quad (\text{Hermitian symmetry})$$

The bar in the above line designates the complex conjugate.

Definition 1.3 A system S of elements of an inner product space is called orthogonal if $(f, g) = 1$ for each $f \in S$, then the system is called orthogonal.

Definition 1.4 Two functions are said to be orthogonal to one another if their inner product is zero, hence a family of functions forms an orthogonal system on an interval (a, b) with a weight function $w(x)$ if for any two distinct members of the family $(\varphi_1, \varphi_2) = 0$.

Definition 1.5 NORM OF POLYNOMIAL

Let $P[X]$ be a linear space of polynomials in indeterminate X and let,
 $11.11 : P[X] \rightarrow \mathbb{R}$

then is called a norm on $P[X]$ if the following axioms are satisfied

- $\|p\| \geq 0$ (positive property)
 - $\|p\| = 0$ if and only if $p = 0$
 - $\|p + q\| \leq \|p\| + \|q\|$ (triangle inequality)
 - $\|\lambda p\| = |\lambda| \|p\|$ (for all $p, q \in P[X], \lambda \in \mathbb{R}$)
- Given a vector u the norm of u is written as

$$\|u\| = \sqrt{(u_1^2 + u_2^2 + \dots + u_n^2)^{1/2}}$$

Basic Properties of Orthogonal Polynomials

Two functions are said to be orthogonal to one another if their inner product is zero, hence a family of functions forms an orthogonal system on an interval (a, b) with a weight function $w(x)$ if for any two distinct members of the family $(\varphi_1, \varphi_2) = 0$. An orthogonal system can be written as a sequence of functions $\{\varphi_n\}_{n=0}^{\infty}$ and the corresponding orthogonal property can be expressed as $(\varphi_m, \varphi_n) = 0$ for $n \neq m$. Assuming that $\{\varphi_n\}$ doesn't contain any null function, then (φ_n, φ_n) is positive for all n and consequently the functions of any finite subset of an orthogonal system are linearly independent. Then the functions $\{\varphi_n\}$ form an orthogonal system if

$$\langle \varphi_n, \varphi_m \rangle = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

Although individually these functions are unimportant, collectively this property of orthogonality with a given weight, is a decisive property that fixes the function uniquely.

System of Orthogonal Functions.

With a specific interval (a, b) (on the real line \mathbb{R}) and a fixed weight function, we can define an inner product for a pair of functions. An inner product may be defined by a Stieltjes integral

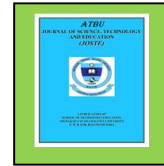
$$(\varphi_1, \varphi_2) = \int_a^b \varphi_1(x)\varphi_2(x)d\mu(x).$$

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where $\mu(x)$ is a non-decreasing function. if $\mu(x)$ is absolutely continuous the the equation above reduces to

$(\varphi_1, \varphi_2) = \int_a^b \varphi_1(x)\varphi_2(x)w(x)d(x)$ where the integral is assumed to exist in Lebesgue's sense. However if $\mu(x)$ is a jump function that is constant except for jumps of the magnitude w_i at $x = x_i$, then it reduces to

$$(\varphi_1, \varphi_2) = \sum_i w_i \varphi_1(x_i) \varphi_2(x_i)$$

the definition for functions of a discrete variable.

Single-Variable Orthogonal Polynomials

Moving from general orthogonal functions to polynomials, we consider a single variable polynomial $P_n(x)$

$$\sum_{j=0}^n a_{n,j} x^j = a_{n,n} x^n + a_{n,n-1} x^{n-1} + \dots + a_{n,1} x + a_{n,0}$$

and for convenience we will deal monic polynomial of order $n(a_{n,n} = 1)$ with some given coefficients $a_{n,0}, a_{n,1}, \dots, a_{n,n}$. What makes these rather simple functions interesting is their orthogonality property, by which they form a family of orthogonal functions i.e a family of polynomials organized according to their degree and within the family, due to the orthogonality to each other. Formally we define an orthogonal polynomial sequence $\{P_n(x)\}^\infty$, with respect to a moment functional L provided for all non-negative integers m and n ,

- $P_n(x)$ is a polynomial of degree n ,
- $L[P_n(x)] = (P_n, P_m)$ where

$$L[P_n(x)] = \int w(x) P_n(x) dx .$$

Application of Orthogonal Polynomial In Relation to Some Fields

The connection of orthogonal polynomials with other branches of mathematics is truly impressive. Without even trying to be complete, we mention continued fractions, operator theory (Jacobi operators), moment problems, analytic functions (Bieberbachs conjecture), interpolation, Pade approximation,

quadrature, approximation theory, numerical analysis, electrostatics, statistical quantum mechanics, special functions, number theory (irrationality and transcendence), graph theory (matching numbers), combinatorics, random matrices, stochastic processes (birth and death processes; prediction theory) data sorting and compression, Radon transform and computer tomography and other areas of application of orthogonal polynomial.

LITERATURE REVIEW

Martin E. Muldon (1989) – Survey on Zero Approximations and Inequalities. Martin E. Muldon conducted an influential survey on methods to find inequalities, monotonicity properties, and approximations for the zeros of orthogonal polynomials and related functions. His approach heavily utilized ordinary differential equations that these polynomials satisfy. Muldon's goal was to describe classic methods for zero approximation, without exploring alternative techniques like recurrence relations and weight functions. His work remains a reference point for understanding traditional approaches to zeroes in orthogonal polynomials.

Milvoje Lukic – Recursion Coefficients and Zero Behavior

Milvoje Lukic expanded the study of orthogonal polynomials through his work on recursion coefficients of polynomials with bounded variation. His focus lay in examining the bounds, zero spacing, and monotonicity of the zeros in relation to parameters. Lukic also applied the Sturm comparison theorem, a fundamental tool for analyzing polynomial behavior, adding depth to the understanding of zero patterns in orthogonal polynomial sequences.

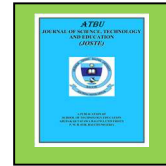
Thomas Jan Stieltjes – Introduction of the Stieltjes Integral and the Moment Problem. Thomas Jan Stieltjes (1856–1894) made a groundbreaking contribution with the Stieltjes integral, a concept that redefined the way mass distribution is represented in terms of an increasing function. This integral was crucial for solving the “moment problem,” a problem of characterizing distributions of orthogonal

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polynomials. Stieltjes' integral has since become foundational for those studying weight functions and measures in the field.

Castell, Filbir, and Forster – *Multivariable Orthogonal Polynomials*. Castell, Filbir, and Forster explored orthogonal polynomials of several variables, laying down a basic theoretical foundation complemented with practical examples. They paid particular attention to polynomials associated with classical weight functions supported on standard domains. Their work remains accessible, requiring only a basic understanding of classical orthogonal polynomials in a single variable and providing explicit formulas for multivariable cases.

Douglas A. Leonard (1982) – Duality and Association Schemes

Douglas A. Leonard's work on duality in orthogonal polynomials introduced the concept of eigenvalues and duality via Delsarte's framework. He reconstructed and characterized Askey-Wilson orthogonal polynomials, demonstrating their duals within P - and Q -polynomial association schemes. His work provides a closed form for the parameters of these polynomials, enabling further research into the dual structures of polynomial association schemes.

Computational Advances – Development of the Maple Library (MOPS)

A major step in computational methods was marked by the development of the Maple library (MOPS), designed for computing various polynomial ensembles such as Jack, Hermite, Laguerre, and Jacobi polynomials. This library supports both symbolic and numerical evaluation of these polynomials, validating algorithmic effectiveness through performance analysis. Such tools have revolutionized the ease with which researchers can approach multivariate orthogonal polynomial computations.

D. Jackson – Orthogonal Sets through Gram-Schmidt Process

D. Jackson contributed by studying properties of orthogonal polynomials in two variables, applying the Gram-Schmidt process to

sequences in a two-dimensional space. His orthogonal sets were derived for specific weight functions, creating a structured approach for polynomials in multiple variables, although he did not extend his work to partial differential equations.

Fast Algorithms for Polynomial Expansions

Several researchers have contributed to the development of fast algorithms in orthogonal polynomial expansions. Alpert and Rokhlin (1991) introduced a complex method for Legendre polynomial expansions using fast multipole techniques. However, due to its complexity, this method has limited application in multivariable contexts. Later, Potts, Steidl, and Tasche (1998) developed a generalized methodology that combines Clenshaw-Curtis quadrature and divide-and-conquer techniques, making it feasible to compute expansions across general orthogonal polynomials.

Recent Innovations – Iserles and Cantero on Legendre Expansions

In 2011, Iserles and Cantero made advancements in computing Legendre polynomial expansions by generalizing the method for settings where the integral measure takes the form $d\mu(x)=(1-x^2)^\alpha dx$ for $\alpha > -1$. This innovation has broadened the scope of applications, allowing for the fast computation of polynomial terms even in complex settings. Emerging Research Needs – Multivariate Special

Function Algorithms

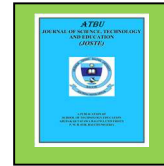
Recent steps have been taken toward developing software to compute multivariate special functions. However, systematic studies, comprehensive handbooks, and application collections remain lacking. The current algorithms mainly address Jack polynomial theories, suggesting a growing field that requires further exploration and organization.

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Frank Nijhoff – Connections between Orthogonal Polynomials and Discrete Systems

Frank Nijhoff's research delved into the relationship between orthogonal polynomials and discrete integrable systems, which involve differential or difference equations solvable by exact methods. He explored how coefficients of orthogonal polynomials are governed by transcendental equations, particularly the Painlevé equations, which unveil complex structures within integrable systems. This comprehensive survey reflects how the field of orthogonal polynomials continues to evolve, blending foundational theories with advanced computational methods. Through these diverse contributions, orthogonal polynomials maintain their relevance across mathematics, physics, and computational fields, underlining both the depth and applicability of this area of study.

ORTHOGONALITY IN POLYNOMIALS

INTRODUCTION

We observe that a truncated Taylor series is really just a polynomial in x and that the only arithmetic operations used to compute it are precisely those that a computer can do. We also see that if $f(x)$ has discontinuities, the Taylor series cannot approximate it over the discontinuity. Polynomials occur frequently in Numerical analysis and some other areas of Science. However we will encounter them in the some instances stated in this chapter, the orthogonality in polynomials and also the 'very' classical orthogonal polynomials, some applications of orthogonal polynomials to mathematics in particular (chapter 4).

One reason for such prevalence is that the only Mathematics needed for their evaluation are addition and multiplication and this fits perfectly. A second reason is the nice behaviour of polynomials: They are everywhere continuous and have derivatives of any order. A famous theorem states that any continuous function can be approximated uniformly over a finite interval by a polynomial!

Certain polynomials are especially useful; many of these have a property called orthogonality. Typical of these are the Chebyshev polynomials. Another set of polynomials that has the orthogonality property is the set of Legendre polynomials, which are involved in a particular good way to integrate a function numerically. This chapter addresses the most relevant properties of orthogonal polynomials, and introduces the transforms associated with them and particularly the orthogonality of Legendre, Bessel and Chebyshev's Polynomials and illustrative examples however not over-emphasizing the 'very' classical orthogonal polynomial.

Definition 3.1 Two polynomials are orthogonal on the close interval $[a, b]$ if their inner product is zero. i.e

$$(f, g) = \int_a^b f(x)g(x)dx = 0$$

The orthogonal basis u_1, u_2, \dots, u_n with $\|u_i\| = 1$ for all i forms an orthogonal basis.

So also we can define orthogonal polynomial with respect to P : the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

Suppose that P is endowed with an inner product.

Definition 3.2 Orthogonal polynomials (relative to the inner product) are polynomials p_0, p_1, p_2, \dots such that $\deg p_n = n$ (p_0 is a nonzero constant) and $(p_n, p_m) = 0$ for $n \neq m$

Orthogonal Polynomials can be obtained by applying the Gram Schmidt orthogonalization process to the basis $1, x, x^2, \dots$

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$$\begin{aligned}
 p_0 &= 1 \\
 p_1(x) &= x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) \\
 p_2(x) &= x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) \\
 &\vdots \\
 p_n(x) &= x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \dots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x)
 \end{aligned}$$

Then p_0, p_1, p_2, \dots are orthogonal polynomials.

Hence $P_{2k}(x)$ contains only even powers of x while $P_{2k+1}(x)$ contains only odd powers of x

$$\begin{aligned}
 p_0(x) &= 1 \\
 p_1(x) &= x \\
 p_2(x) &= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = x^2 - \frac{1}{3} \\
 p_3(x) &= x^3 - \frac{\langle x^3, x \rangle}{\langle x, x \rangle} x = x^3 - \frac{3}{5} x \\
 \langle x^n, x^m \rangle &= 0 \text{ if } m + n \text{ is odd.}
 \end{aligned}$$

Hence $p_{2k}(x)$ contains only even powers of x while $p_{2k+1}(x)$ contains only odd powers of x .
 p_0, p_1, p_2, \dots are called the **Legendre Polynomials**.

Instead of normalization, the orthogonal polynomials are subject to **Standardization**. The standardization for the Legendre Polynomials is $p_n(1) = 1$. In particular,

$$p_0(x) = 1, p_1(x) = x, p_2(x) = \frac{1}{2}(3x^2 - 1), p_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Problem 1

Obtain the term $p_4(x)$ from the above example (3.2)

Solution

Let $p_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

We know that $p_4(1) = 1$ and $\langle p_4, x^k \rangle = 0$ for $0 \leq k \leq 3$.

$$p_4(1) = a_4 + a_3 + a_2 + a_1 + a_0, \quad \langle p_4, 1 \rangle = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0,$$

$$\langle p_4, x \rangle = \frac{2}{3}a_3 + \frac{2}{5}a_1, \quad \langle p_4, x^2 \rangle = \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0, \quad \langle p_4, x^3 \rangle = \frac{2}{7}a_3 + \frac{2}{5}a_1$$

$$a_4 + a_3 + a_2 + a_1 + a_0 = 1$$

$$\frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0$$

$$\frac{2}{5}a_3 + \frac{2}{5}a_1 = 0$$

$$\frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0$$

$$\frac{2}{7}a_3 + \frac{2}{5}a_1 = 0$$

Solve simultaneously

$$\frac{2}{5}a_3 + \frac{2}{5}a_1 = 0$$

$$\frac{2}{7}a_3 + \frac{2}{5}a_1 = 0$$

$$a_1 = a_3 = 0$$

Therefore, it implies that,

$$\Rightarrow a_4 = \frac{35}{8}, a_2 = -\frac{30}{8}, a_0 = \frac{3}{8}$$

Thus, $p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

MONIC ORTHOGONAL POLYNOMIAL

Definition 3.3 Polynomials in a set are said to be monic orthogonal polynomials if they are orthogonal, monic and their norms are strictly positive.

Theorem 3.1 For monic orthogonal polynomials, there exists sequences of coefficients $\alpha_k, k = 1, 2, \dots$ and $\gamma_k, k = 0, 1, 2, \dots$ such that

$$P_{k+1}(\lambda) = (\lambda - \alpha_{k+1})P_k(\lambda) - \gamma_k P_{k-1}(\lambda), \quad k = 0, 1, \dots$$

$$P_{-1}(\lambda) \equiv 0, P_0 \equiv 1$$

Proof 3.1 A set of monic orthogonal polynomials P_j is linearly independent. Any polynomial P of degree K can be written as:

$$P = \sum_{j=0}^K \omega_j P_j$$

for some real numbers ω_j .

$P_{k+1} - \lambda P_k$ is of degree $\leq K$

$$P_{k+1} - \lambda P_k = -\alpha_{k+1}P_k - \gamma_k P_{k-1} + \sum_{j=0}^{k-2} \delta_j P_j \quad (3.1)$$

Taking the inner product of the equation above with P_k

$$\langle \lambda P_k, P_k \rangle = \alpha_{k+1} \langle P_k, P_k \rangle$$

Multiplying by P_{k-1}

$$\langle \lambda P_k, P_{k-1} \rangle = \gamma_k \langle P_{k-1}, P_{k-1} \rangle$$

But, using the equation for the degree $k - 1$

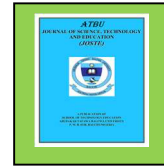
$$\langle \lambda P_k, P_{k-1} \rangle = \langle P_k, P_{k-1} \rangle = \langle P_k, P_k \rangle$$

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We multiply equation 3.3 with P_j , $J < k - 1$
 $(\lambda P_k, P_j) = \sigma_j(P_k, \lambda P_j)$
 The left-hand side of the last equation vanishes.

For this, the property $(\lambda P_k, P_j) = (P_k, \lambda P_j)$ is crucial.
 Since λP_j is of degree j , the left hand side is 0
 and it implies $\delta_j = 0$, $j = 0, 1, \dots, k - 2$

ORTHOGONALITY WITH RESPECT TO A WEIGHT FUNCTION

Definition 3.4 If

$$\int_a^b \varphi_m(x)\varphi_n\omega(x)dx = \delta(mn)(3.2)$$

where $\omega(x) \geq 0$, we often say that the set $\varphi_k(x)$ is orthogonal with respect to the density function or weight function $\omega(x)$. In such case the set $\varphi_m(x) = \phi_m(x) = \sqrt{\omega(x)\phi_m(x)}$, $m = 1, 2, 3, \dots$, is an orthogonal set in (a, b) .
 A necessary and sufficient condition for having a positive definite inner product is that determinants of the matrices are positive

$$\det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{k-1} \\ \mu_1 & \mu_2 & \dots & \mu_k \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k-1} & \mu_k & \dots & \mu_{2k-2} \end{pmatrix} > 0, k = 1, 2 \quad (3.3)$$

where μ_i are the moments of definition.

Orthogonality Of Chebyshev's Polynomial

Definition 3.5

The Chebyshev's differential equation
 $(1 - x^2)y'' - xy' + n^2y = 0 \quad (3.8)$

can also be written in Sturm-liouville form

$$\frac{d}{dx}[(1 - x^2)y'] + n^2(1 - x^2)^{-\frac{1}{2}}y = 0 \quad (3.9)$$

Let y_m and y_n , $m \neq n$ be the chebyshev's polynomials satisfying equation (3.9), we have

$$\frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_m] + m^2(1 - x^2)^{-\frac{1}{2}}T_m \quad (3.10)$$

and

$$\frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_n] + n^2(1 - x^2)^{-\frac{1}{2}}T_n \quad (3.11)$$

where $y_n = T_n$ and $-y_m = -T_m$ respectively.

By multiplying equation (3.10) by y_n and equation (3.11) by $-y_m$

$$\frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_m]T_n + m^2(1 - x^2)^{-\frac{1}{2}}T_mT_n = 0 \quad (3.12)$$

$$-\frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_n]T_m - n^2(1 - x^2)^{-\frac{1}{2}}T_mT_n = 0 \quad (3.13)$$

Add equation (3.12) and (3.13)

$$\frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_m]T_n - \frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_n]T_m + (m^2 - n^2)(1 - x^2)^{-\frac{1}{2}}T_mT_n = 0$$

$$(n^2 - m^2)(1 - x^2)^{-\frac{1}{2}}T_mT_n = \frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_m]T_n - \frac{d}{dx}[(1 - x^2)^{-\frac{1}{2}}T_n]T_m$$

Integrating both sides from $(-1, 1)$,

$$(n^2 - m^2) \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} T_m T_n dx = \int_{-1}^1 \frac{d}{dx} [(1 - x^2)^{-\frac{1}{2}} T_m] T_n - \int_{-1}^1 \frac{d}{dx} [(1 - x^2)^{-\frac{1}{2}} T_n] T_m dx$$

$$(n^2 - m^2) \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} T_m T_n dx = [(1 - x^2)^{\frac{1}{2}} T_m] T_n - [(1 - x^2)^{\frac{1}{2}} T_n] T_m \Big|_{-1}^1$$

$$n^2 - m^2 \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} T_m T_n dx = 0$$

Recall: $n \neq m$

$$n^2 - m^2 \neq 0$$

It implies that, $\int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} T_m(x) T_n(x) dx = 0$

Therefore, T_m and T_n are orthogonal on $(-1, 1)$ with respect to the weight function

$$\omega(x) = (1 - x^2)^{-\frac{1}{2}}$$

Example 3.4 An example to illustrate the Chebyshev Polynomials:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), k = 1, 2, \dots$$

are orthogonal with respect to two inner products:

$$\langle g, h \rangle = \int_{-1}^1 \frac{g(x)h(x)}{\sqrt{1 - x^2}} dx$$

In this case,

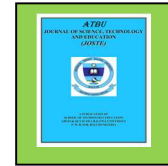
$$\langle T_i, T_j \rangle = \int_{-1}^1 \frac{T_i(x)T_j(x)}{\sqrt{1 - x^2}} dx$$

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ORTHOGONALITY OF BESSEL'S POLYNOMIAL

The Bessel's differential equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad (3.14)$$

can be re-written in a Sturm Liouville form

$$\frac{d}{dx} [x(xy') + (x^2 - n^2)y] = 0 \quad (3.15)$$

Let $J_m(x)$ and $J_n(x)$, ($n \neq m$) be the Bessel Polynomial satisfying equation (3.15), we have

$$\frac{d}{dx} [x(xJ_n)'] + (x^2 - n^2)J_n = 0 \quad (3.16)$$

and

$$\frac{d}{dx} [x(xJ_m)'] + (x^2 - m^2)J_m = 0 \quad (3.17)$$

respectively.

Multiply equation (3.16) by J_m and equation (3.17) by $-J_n$, we have

$$\frac{d}{dx} [x(xJ_n)']J_m + (x^2 - n^2)J_n J_m = 0 \quad (3.18)$$

$$\frac{d}{dx} [-x(xJ_m)']J_n - (x^2 - m^2)J_n J_m = 0 \quad (3.19)$$

Add the equation (3.18) and (3.19), we have

$$\frac{d}{dx} [x(xJ_n)']J_m - x(xJ_m)']J_n + (x^2 - n^2)J_n J_m - (x^2 - m^2)J_n J_m = 0$$

$$\frac{d}{dx} [x(xJ_n)']J_m - x(xJ_m)']J_n + J_n J_m [(x^2 - n^2) - (x^2 - m^2)]$$

$$\frac{d}{dx} [(xJ_n)']J_m - x(J_m)']J_n + xJ_n J_m [(m^2 - n^2)]$$

$$(m^2 - n^2)xJ_n J_m = \frac{d}{dx} [x(J_n)']J_m - x(J_m)']J_n$$

$$(m^2 - n^2) \int_0^1 xJ_n J_m = 0$$

$$(m^2 - n^2) \int_0^1 xJ_n(x)J_m(x)dx = 0$$

$$\text{Since, } m^2 - n^2 \neq 0$$

$$\text{Therefore implies that } \int_0^1 xJ_n(x)J_m(x)dx.$$

THE 'Very' CLASSICAL ORTHOGONAL POLYNOMIALS

In the modern theory the following are referred to as the very classical orthogonal polynomials

1. Hermite polynomials
2. Laguerre (Generalized) polynomials
3. Jacobi or Hypergeometric polynomials

(of which, these are all characterized by their different weight functions and integration intervals), where it is important to highlight the

distinction between these and other orthogonal polynomials. But where does that distinction lie?

The classical orthogonal polynomials can be defined as those orthogonal polynomials satisfying the properties:

1. $P_n(x)$ is a system of orthogonal polynomials,

2. $P_n(x)$ satisfies a differential equation of the form

$$A(x)y'' + B(x)y' + \lambda_n y = 0$$

where $A(x)$ and $B(x)$ are independent of n and λ_n is independent of x ;

3. There is a general Rodrigue's Formula

$$P_n(x) = \frac{1}{k_n \omega(x)} \frac{d}{dx^n} (\omega(x) X^n)$$

where k_n is a constant and X is a polynomial in X , whose coefficients are independent of n .

Conversely any of these three properties characterizes the classical orthogonal polynomials in the sense that any system of orthogonal polynomials which has one of these properties can be reduced to a classical system.

However, in recent times there have been number of families that satisfy these conditions, but are not called classical. Thus, we refer to those mentioned above as the very classical orthogonal polynomials.

Looking more closely at the properties that these families have, it can be shown that they all have a 'generating function, a recursion and differential relation' (which can be combined to give a second order ODE) and a Rodrigue's formula. These equations are easier to derive for the Hermite and Legendre polynomials and more difficult for the other families with the Jacobi polynomials providing large equations, which although solvable, require some mathematical tricks and techniques. Families of orthogonal polynomials have a lot of beautiful mathematics objects to study.

The generating function for the family Hermite can be used to calculate the other three properties, i.e, the recurrence and differential relations and the Rodrigue's formula, and is also

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used useful in proving the orthogonality of family. To emphasize this point the Hermite family will be defined through the use of its generating function. However it is not the generating function alone that shares this property, since the Rodrigues' formula can also be used to define the recursion and differential relations, and prove the orthogonality.

APPLICATIONS OF ORTHOGONALITY

There are a number of areas in which orthogonal polynomials have been used extensively. Among the traditional applications are orthogonal expansions, least squares approximation, interpolation, and numerical quadrature. Somewhat less known are the connections of orthogonal polynomials with Padé approximation, when the underlying power series has moments as coefficients. More recent application include those in probability theory e.g birth and death processes, coding theory, relaxation methods in numerical linear algebra. Szegő's theorem on the asymptotic behaviour of Toeplitz determinants also has important applications in statistical mechanics. Here, considering only some recent simple applications to interpolation orthogonal polynomials, linear and vector spaces, since all of these involve interpolation, we begin with basis of interpolating and interpolation to orthogonal polynomials.

INTERPOLATION

What is Interpolation?

Imagine that there is an unknown function $f(x)$ for which someone supplies you with its (exact) values at $(n + 1)$ distinct points $x_0 < x_1 < \dots < x_n$, i.e, $f(x_0), \dots, f(x_n)$ are given .

The interpolation problem is to construct a function $Q(x)$ that passes through these points, i.e, to find a function $Q(x)$ such that the interpolation requirements.

$$Q(x_j) = f(x_j), \quad 0 \leq j \leq n, \quad (4.1)$$

are satisfied. One easy way of obtaining such a function, is to connect the given points with straight lines. While this is a legitimate solution of

the interpolation problem, usually (though not always) we are interested in a different kind of solution, e.g. a smoother function. We therefore always specify a certain class of functions from which we would like find one that solves the interpolation problem. For example, we may look for a function $Q(x)$ that is polynomial, $Q(x)$. Alternatively, the function $Q(x)$ can be trigonometric function or a piecewise-smooth polynomial, and so on.

THE INTERPOLATION PROBLEM

We begin our study with the problem of polynomial interpolation: Given $n + 1$ distinct points x_0, \dots, x_n , we seek a polynomial $Q_n(x)$ of the lowest degree such that the following interpolation conditions are satisfied:

$$Q_n(x_j) = f(x_j), \quad j = 0, 1, \dots, n.$$

Note that we do not assume any ordering between the points x_0, \dots, x_n , as such an order will make no difference. If we do not limit the degree of the interpolation polynomial it is easy to see that there are infinitely many polynomials that interpolate the data. However, limiting the degree of $Q_n(x)$ to be $\deg Q_n(x) \leq n$, singles out precisely one interpolant that will do the job. For example, if $n = 1$, there are infinitely many interpolate $(x_0, f(x_0))$ and $(x_1, f(x_1))$. However there is only one polynomial $Q_n(x)$ with $\deg Q_n(x) \leq n$ that does the job. This result is formally stated in the following theorem.

Theorem 4.2.1

If $x_0, \dots, x_n \in \mathbb{R}$ are distinct, then for any $f(x_0), \dots, f(x_n)$ there exists a unique polynomial $Q_n(x)$ of degree $\leq n$ such that the interpolation conditions (4.2) are satisfied

Proof

We start with the existence part and prove the result by induction. For example $n = 0$, $Q_0 = f(x_0)$. Suppose that Q_{n-1} is a polynomial of degree $\leq n - 1$, and suppose also that

$$Q_{n-1}(x_j) = f(x_j), \quad 0 \leq j \leq n - 1$$

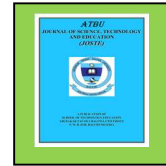
Let now construct from $Q_{n-1}(x)$ a new polynomial, $Q_n(x)$, in the following way:

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$$Q_n(x) = Q_{n-1}(x) + c(x - x_0) \cdots (x - x_{n-1}). \quad (4.2)$$

The constant c in (4.2) is yet to be determined. Clearly, the construction of $Q_n(x)$ implies that $\deg(Q_n(x)) \leq n$. (Since we might end up with $c = 0$, $Q_n(x)$ could actually be degree that is less than n .) In addition, the polynomial $Q_n(x)$ satisfies the interpolation requirements $Q_n(x_j) = f(x_j)$ for $0 \leq j \leq n - 1$. All that remains is determine the constant c in such a way the last interpolation condition, $Q_n(x_n) = f(x_n)$ is satisfied, i.e.,

$$Q_n(x) = Q_{n-1}(x_n) + c(x_n - x_0) \cdots (x_n - x_{n-1}). \quad (4.3)$$

The condition (4.3) implies that c should be defined as:

$$c = \frac{f(x_n) - Q_{n-1}(x_n)}{\prod_{j=0}^{n-1} (x_n - x_j)}, \quad (4.4)$$

and we are done with the proof of existence. As for uniqueness, suppose that there are two polynomials $Q_n(x)$, $P_n(x)$ of degree $\leq n$ that satisfy the interpolation conditions (4.2). Define a polynomial $H_n(x)$ as the difference $H_n(x) = Q_n(x) - P_n(x)$.

The degree of $H_n(x)$ is at most n which means that it can have at most n zeros (unless it is identically zero). However, since both $Q_n(x)$ and $P_n(x)$ satisfy all the interpolation requirements (4.2), we have:

$$H_n(x) = Q_n(x) - P_n(x) = 0, \quad 0 \leq j \leq n,$$

which means that $H_n(x)$ has $n + 1$ distinct zeros. This contradiction can be resolved only if $H_n(x)$ is zero polynomial, i.e., $P_n(x) \equiv Q_n(x)$, and uniqueness is established.

Newton's Form of the Interpolation Polynomial

One good thing about the proof of Theorem 4.1 is that it is constructive. In other words, we can use the proof to write down a formula for the interpolation polynomial. We follow the procedure given by (4.3) for reconstructing the

interpolation polynomial. We do it in the following way:

• Let

$$Q_0(x) = a_0, \text{ where } a_0 = f(x_0).$$

• Let

$$Q_1(x) = a_0 + a_1(x - x_0).$$

following (4.4) we have

$$Q_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1}) \\ = a_0 + \sum_{j=1}^n a_j \prod_{k=0}^{j-1} (x - x_k). \quad (4.5)$$

The coefficients a_j in (4.5) are given by

$$a_0 = f(x_0), \\ a_j = \frac{f(x_j) - Q_{j-1}(x_j)}{\prod_{k=0}^{j-1} (x_j - x_k)}, \quad 1 \leq j \leq n \quad (4.6)$$

We refer to the interpolation polynomial when written in the form (4.5)-(4.6) as the Newton form of the interpolation polynomial. As we shall see below, there are various ways of writing the interpolation polynomial. The uniqueness of the interpolation as guaranteed by Theorem 4.1 implies that we will only be rewriting the same polynomial in different ways.

4.3.1 Example

The Newton form of the polynomial that interpolates $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$Q_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Example

The Newton form of the polynomial that interpolates the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is:

$$Q_2(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{f(x_2) - [f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)]}{(x_2 - x_0)(x_2 - x_1)}(x - x_0)(x - x_1).$$

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The Lagrange Form of the Interpolation Polynomial

The form of the interpolation polynomial that was used in

$$Q_n(x) = \sum_{k=0}^n b_k x^k,$$

assumed a linear combination of polynomials of degrees $0, \dots, n$, in which the coefficients were unknown. In this section we take a different approach and assume that the interpolation polynomial is given as a linear combination of $n + 1$ polynomials of degree n . This time, we set the coefficients as the values, $f(x_j)$ for $j=0, \dots, n$, while the unknowns are the polynomials. We thus let

$$Q_n(x) = \sum_{j=0}^n f(x_j) l_j^n(x), \quad (4.7)$$

where $l_j^n(x)$ are $n + 1$ polynomials of degree $\leq n$. We use two indices in these polynomials: the subscript j enumerates $l_j^n(x)$ from 0 to n and the superscript n is used to remind us that the degree of $l_j^n(x)$ is n . Note that in this particular case, the polynomials $l_j^n(x)$ are precisely of degree n (and not $\leq n$). However, $Q_n(x)$, given by (3.15) may have a lower degree. In either case, the degree of $Q_n(x)$ is n at the most. We now require that $Q_n(x)$ satisfies the interpolation conditions

$$Q_n(x_i) = f(x_i), \quad 0 \leq i \leq n. \quad (4.8)$$

By substituting x_i for x in (4.7) we have

$$Q_n(x_i) = \sum_{j=0}^n f(x_j) l_j^n(x_i), \quad 0 \leq i \leq n.$$

In view of (4.8) we may conclude that $l_j^n(x)$ must satisfy

$$l_j^n(x_i) = \delta_{ij}, \quad i, j = 0, \dots, n, \quad (4.9)$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Each polynomial $l_j^n(x)$ has $n + 1$ unknown coefficients. The conditions (4.9) provide exactly $n+1$ equations that the polynomials $l_j^n(x)$ must satisfy and these equations can be solved in order to determine all $l_j^n(x)$'s. Fortunately there is a shortcut. An obvious way of constructing polynomials $l_j^n(x)$ of degree $\leq n$ that satisfy (4.9) is the following:

$$l_j^n(x) = \frac{(x-x_0) \dots (x-x_{j-1})(x-x_{j+1}) \dots (x-x_n)}{(x_j-x_0) \dots (x_j-x_{j-1})(x_j-x_{j+1}) \dots (x_j-x_n)}, \quad 0 \leq j \leq n \quad (4.10)$$

The uniqueness of the interpolation polynomial of degree $\leq n$ given $n + 1$ distinct interpolation points implies that the polynomials $l_j^n(x)$ are given by (4.9) are the only polynomials of degree $\leq n$ that satisfy (4.9).

Note that the demonstrator in (4.10) does not vanish since we assume that all interpolation points are distinct. $j = 0, \dots, n$

Example 4.4.1

We are interested in finding the Lagrange form of the interpolation polynomial that interpolates two points: $(x_0, f(x_0))$ and $(x_1, f(x_1))$. We know that the unique interpolation polynomial through these two points in the line that connects the two points. Such a line can be written in many different forms. In order to obtain the Lagrange form we let

$$l_0^1(x) = \frac{x - x_1}{x_0 - x_1}, \quad l_1^1(x) = \frac{x - x_0}{x_1 - x_0}.$$

The desired polynomial is therefore given by the familiar formula

$$Q_1(x) = f(x_0)l_0^1(x) + f(x_1)l_1^1(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0}.$$



Example 4.4.2

This time we are looking for the Lagrange form of the interpolation polynomial, $Q_2(x)$, that interpolates three points: $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Unfortunately, the Lagrange form of the interpolation polynomial does not let us use the interpolation polynomial through the first two points, $Q_1(x)$, as a building block for $Q_2(x)$. This means that we have to compute all the polynomials $l^n(x)$ from scratch. We start with

$$l_0^2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)},$$

$$l_1^2(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)},$$

$$l_2^2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The interpolation polynomial is therefore given by

$$\begin{aligned} Q_2(x) &= f(x_0)l_0^2(x) + f(x_1)l_1^2(x) + f(x_2)l_2^2(x) \\ &= f(x_0)\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2)\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}. \end{aligned}$$

It is easy to verify that indeed $Q_2(x_j) = f(x_j)$ for $j = 0, 1, 2$, as desired.

Remark 4.4.3

- One instance where the Lagrange form of the interpolation polynomial may seem to be advantageous when compared with the Newton form is when there is a need to solve several interpolation problems, all given at the same interpolation points x_0, \dots, x_n but with different values $f(x_0), \dots, f(x_n)$. In this case, the polynomials $l_n(x)$ are identical for all problems since they depend only on the points but not on the values of the function at these points. Therefore, they have to be constructed only once.
- For future reference we note that the coefficient of x^n in the interpolation polynomial $Q_n(x)$ is

$$\sum_{j=0}^n \frac{f(x_j)}{\prod_{k=0, k \neq j}^n (x_j - x_k)}$$

ORTHOGONALITY IN LINEAR SPACES

By $K[x]$ we denote the linear space of polynomials of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad (4.12)$$

where $a_k \in K$ are from a field (for example, \mathbb{C} or \mathbb{R}) and $n \in \mathbb{N} = 0, 1, 2, \dots$ hence, the elements of $\mathbb{C}[x]$ are polynomials with complex coefficients, which we also consider as maps $P: \mathbb{R} \rightarrow \mathbb{C}, x \rightarrow p(x)$ from the set of real numbers into the set of complex numbers. If a_n in (3.1.1) is different from zero, then $\deg p = n$ is called degree of the polynomial the leading coefficient of this polynomial. The linear subspace of $\mathbb{C}[x]$ of all polynomials of degree less than n will be denoted by $\mathbb{C}_n[x], n \in \mathbb{N} = [1, 2, \dots]$

For any $n \in \mathbb{N}$, the system $M_k(x) = x^k, k = 0, 1, \dots, n-1$, forms a basis in $\mathbb{C}_n[x], n \in \mathbb{N} = [1, 2, \dots]$

For any $n \in \mathbb{N}$, the system $M_k(x) = x^k, k = 0, 1, \dots, n-1$, forms a basis in $\mathbb{C}_n[x]$, which means that the representation of $p(x)$ from $\mathbb{C}_n[x]$ is unique. This also means that the each finite subsystem of the system $(M_n)^\infty$ is linearly independent, because of which we also call this infinite linearly independent. As an example, let us define on $\mathbb{C}[x]$ a scalar product, also called inner product.

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx, p, q \in \mathbb{C}[x] \quad (4.13)$$

Applying the Schmidt's orthogonalization method to the system $(M_n)^\infty$, then we get a system $(p_n)^\infty$ of polynomials $P_n(x)$ with the properties $\deg P_n = n$,

$$\text{span } p_0, \dots, p_{n-1} = \mathbb{C}_n[x], n \in \mathbb{N}$$

With $\text{span } P_0, \dots, P_{n-1}$ we refer to the set of all linear combinations of the polynomials $P_j, j = 0, 1, \dots, n-1$. Since this set is equal to $\text{span } M_0, \dots$



M_{n-1} , it is easily seen that for $k \neq j$, the orthogonality relations (3.3) are equivalent to $(M_n, P_n) = 0, K = 0, \dots, n - 1 \in N$

ORTHOGONALITY IN VECTOR SPACES

Definition 4.1 Two vectors A and B are called orthogonal (perpendicular) if $A \cdot B = 0$ or $A_1B_1 + A_2B_2 + A_3B_3 = 0$, where $A = A_1i + A_2j + A_3k$ and $B = B_1i + B_2j + B_3k$.

Although not geometrically or physically obvious, these ideas can be generalized to include vectors with more than three components. In particular we can think of a function, say $A(x)$, as being a vector with an infinitely of components (i.e an infinite-dimensional vector), the value of each component being specified by substituting a particular value of x taken from some interval (a, b) . It is natural in such case to define two polynomials $A(x)$ and $B(x)$, as orthogonal in (a, b) if

$$\int_a^b A(x)B(x)dx = 0 \tag{4.14}$$

The left side of equations is often called the scalar product of $A(x)$ and $B(x)$

A vector A is called a unit vector if its magnitude is unity, i.e if $A \cdot A = A^2 = 1$. Extending the concept, we say that function $A(x)$ is called a unit vector or normalized in (a, b) if

$$\int_a^b (A(x))^2 dx = 1$$

From the above, it is clear that we can consider a set of functions $\phi_k(x), k = 1, 2, 3$ having the properties

$$\int_a^b \phi_m(x)\phi_n(x)dx = 0, m \neq n \tag{4.15}$$

$$\int_a^b \phi_m(x)^2 dx = 1 \tag{4.16}$$

Each member of the set is orthogonal to every other member of the set and is also normalized.

The equations (4.15) and (4.16) can be summarized by writing

$$\int_a^b \phi_m(x)\phi_n(x)dx = \delta_{(mn)}$$

where $\delta_{(mn)}$, called Kronecker's symbol, is defined as 0 if $m \neq n$ and 1 if $m = n$

ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

Let μ be a measure supported on the unit circle and denote by $P = C[Z]$ the linear space of polynomials with complex coefficients and by P_n the subspace of polynomials with degree less or equal to n . $\Lambda := C[Z, Z^{-1}]$ denote the complex vector spaces of Laurent polynomials and $\Lambda_{m,n} := \text{span}\{Z^m, Z^{m+1}, \dots, Z^n\}$ for $m \leq n, m, n \in \mathbb{Z}$. We consider the inner product $\langle zp(z), zq(z) \rangle_\mu = \langle p(z), q(z) \rangle_\mu, p, q \in \Lambda$ (4.19)

where π is the complex unit circle. It is clear that it satisfies the property

$$\langle zp(z), zq(z) \rangle_\mu = \langle p(z), q(z) \rangle_\mu \quad \forall p, q \in \mu \tag{4.20}$$

Applying the Gram-Schmidt orthogonalization procedure over the canonical basis $\{Z^n\}_{n \geq 0}$, we obtain the corresponding polynomials on the unit circle (OPUC).

Application of Orthogonal Polynomial in Sciences and Engineering

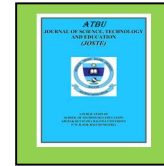
In recent times, the modern Engineering applications need a more thorough knowledge of Mathematics, especially of applied one. Special functions are one of such areas of Mathematics. However, orthogonal polynomials are very important special functions with many applications in Engineering and Sciences. These polynomials appear in many areas of applications in Mathematics and in Physics as approximation theory, Probability theory, Partial differential equations for heat conduction, quantum mechanics, and electric circuit theory, signals processing theory, control theory, statistical communication theory, problems in mechanics and others. Besides, orthogonal polynomials appear in special functions such as in quantum mechanics in Physics as well as some

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Mathematics course such as ordinary differential equations and partial differential equations. They equally become useful in combinatorics, Number Theory, Complex Analysis, and Numerical Analysis.

CONCLUSION

The use of Gram Schmidt and Sturm-Liouville's has proved useful in the generation of orthogonality of the special cases and however the study has been able to address the most relevant properties of orthogonal polynomials, so also the transforms associated with them and particularly the orthogonality of Legendre, Bessel and Chebyshev's polynomials which are involved in a particular good way to integrate a function numerically.

The importance of some of its application in Mathematics which includes: the orthogonality of polynomial in Interpolation, the Newton's and Lagrange form of interpolation, Linear spaces, Vector spaces and Unit circles shows a good blend between abstract and pure Mathematics in the study investigated.

In summary, the study shows that the special cases of polynomials which includes the Chebyshev, Legendre and Gegenbauer polynomials have a good numerical properties, and they are mostly used which evolve through Mathematics, Mathematical Physics and other wide varieties of areas of sciences. However, the figure below is a tabular representation of the three-polynomial worked on together with those briefly discussed with their symbols, weight functions and interval.

polynomials	$P_n(x)$	$w(x)$	Interval
Bessel	$J_n(x)$	x	$(0, 1)$
Chebyshev	$T_n(x)$	$2 \frac{-1}{(1-x)^2}$	$[-1, 1]$
Gegenbauer	$K_n(x)$	$2 \alpha^{-1} \frac{1}{(1-x)^2}$	$[-1, 1]$
Hermite	$H_n(x)$	e^{-x^2}	$(-\infty, \infty)$
Jacobi	$P_n(x)$	$\alpha \beta \frac{1}{(1-x)(1+x)}$	$(-1, 1)$
Legendre	$P_n(x)$	1	$(-1, 1)$

Laguerre	$V_n(x)$	e^{-x}	$[0, \infty]$
Laguerre (assoc)	$V_m(x)$	$x^k e^{-x}$	$[0, \infty]$

RECOMMENDATIONS

So far, we have seen the use of orthogonal polynomials can help us solve the normal equations which arise in discrete and continuous interpolation problems, without the need for expensive and numerically difficult matrix inversions. The ideas and techniques we developed —i.e Gram-Schmidt orthogonalization with respect to a weight function over any interval have applications far beyond interpolation problems, vector and linear spaces.

Orthogonal polynomials, Orthogonalization and its Applications in Mathematics and Sciences and other areas by virtue of its multi-disciplinary nature should be studied more comprehensively in the undergraduate program. More research work should be carried out on other types of orthogonal polynomials as well as the comprehensive study of the applications of orthogonal polynomials in future.

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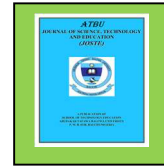
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