
BAYESIAN ESTIMATION OF SCALE PARAMETER OF THE LOG-LOGISTIC DISTRIBUTION UNDER THE ASSUMPTION OF CHI-SQUARE AND MAXWELL PRIORS

By

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ABSTRACT

In this paper, the estimation of the scale parameter of log-logistic distribution was addressed under the assumption of chi-square and Maxwell priors wherein the Bayes estimates and posterior risks were derived under squared error loss function (SELF) and precautionary loss function (PLF). Furthermore, a simulation study was undertaken using the Markov-chain Monte Carlo (MCMC) simulation technique in order to assess the performance of the assumed prior distributions and loss functions. Among the two priors used in this study, chi-square prior appears to yield lower posterior risks thereby producing the best estimate and when compared with the estimates produced under the assumption of non-informative priors (uniform and Jeffrey's), we realized that indeed the assumption of informative priors yields better estimates and overall the precautionary loss function performs better for the estimation of the parameter of interest.

Keywords: Log-logistic Distribution, Bayes Estimates, Posterior risks, Prior Distribution, Loss function, Markov-Chain Monte Carlo Simulation, squared error loss function (SELF), Precautionary loss function.

INTRODUCTION

In economics, the log-logistic distribution is also known as the Fisk distribution, it is a continuous distribution that is applied in fields such as: hydrology, economics, biology and a lot more. The log-logistic distribution can also be alternatively used in place of log-normal distribution (Akhter and Khan, 2014) and it is a derivation of the logistic distribution as stated in (Aryal, 2013). In practice, if T is a random variable having a log-logistic distribution, then $Y = \log T$ is a Random variable that follows a logistic distribution (Santana *et al.*, 2012).

Bayesian inference is an estimation technique that uses a prior information or knowledge about the distribution of interest in combination with the available information (in form of likelihood) of that distribution to

estimate the parameter(s) of such distribution. This technique adopts the Baye's theorem to combine the prior distribution and likelihood function to obtain the posterior distribution with respect to the parameter of interest, after-which inferences can be made about that parameter.

Several authors have adopted the Bayesian technique of estimation to estimate the parameters of so many distributions among which Nasir *et. al.* in 2015 used the technique to estimate the scale parameter of log-logistic distribution under the assumption of non-informative priors (uniform and Jeffrey's). Hence in this work we used the same technique but with the assumption of informative priors (Maxwell and Chi-square) to estimate the scale parameter of log-logistic distribution.

MATERIALS AND METHOD

Bayesian setup

The likelihood function of a distribution is denoted as $p(x|r)$, x here denotes the vector of the observed data $x = \{x_1, x_2, \dots, x_n\}$ in

this case it is assumed that our data is a random sample (iid) and hence $p(x|r) = \prod_{i=1}^n p(x_i|r)$. Thus, all inferences on α are given by the posterior distribution:

$$p(r|x) = \frac{p(x|r)f(r)}{\int_{-\infty}^{\infty} p(x|r)f(r)dr} \tag{1}$$

$$p(r|x) \propto p(x|r)f(r)$$

Model and Likelihood function

The probability density function(PDF) of log-logistic distribution is given by:

$$f(x;r,s) = \frac{r x^{r-1}}{s^r \left[1 + \left(\frac{x}{s} \right)^r \right]^2} \quad \text{where } x, r, s > 0 \tag{2}$$

The cumulative distribution function(CDF) is given by:

$$F(x;r,s) = \frac{1}{1 + \left(\frac{x}{s} \right)^r} \quad \text{where } x, r, s > 0 \tag{3}$$

Here r is the scale parameter while s is the shape parameter.

The likelihood function of the distribution is as follows;

$$L(x;r,s) = \frac{r^n \ell^{\sum_{i=1}^n \ln x_i^{r-1}}}{\ell^{nr \ln s} \ell^{2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right]}}$$

$$L(x;r,s) = r^n \times \ell^{\sum_{i=1}^n \ln x_i^{r-1}} \times \ell^{-nr \ln s} \times \ell^{-2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right]}$$

$$L(x;r,s) = r^n \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right]} \tag{4}$$

Prior distribution

1. Chi-square prior is assumed to be;

$$p(r) \propto r^{k-1} \ell^{\frac{r^2}{2}} \quad \text{where } k, r > 0 \tag{5}$$

2. Maxwell prior is assumed to be;

$$p(r) \propto r^2 \ell^{-\frac{r^2}{a}} \quad \text{where } a, r > 0 \tag{6}$$

Loss and Risk function

In this study, we applied the squared error loss function (SELF) which is symmetrical and precautionary loss function (PLF) that is asymmetrical.

1. The squared error loss function is given by;

$$L(\hat{r}_{sl}, r) = (\hat{r}_{sl} - r)^2 \tag{7}$$

Where \hat{r}_{sl} denotes the Bayes estimator of α relative to the SELF and it's given by:

$$\hat{r}_{sl} = E(r | x) \tag{8}$$

The risk function of the SELF is:

$$R\hat{r}_{sl} = E(r^2) - [E(r)]^2 \tag{9}$$

1. Precautionary loss function introduced by Norstrom (1996) is an asymmetric loss function which is given as:

$$L(\hat{r}_{pl}, \alpha) = \frac{(\hat{r}_{pl} - r)^2}{\hat{r}_{pl}} \tag{10}$$

Where \hat{r}_{pl} denotes the Bayes estimator of α relative to the PLF and it is given by:

$$\hat{r}_{pl} = \sqrt{E(r^2 | x)} \tag{11}$$

Hence the risk associated with the PLF is:

$$R\hat{r}_{pl} = 2 \left[\sqrt{E(r^2)} - E(r) \right] \tag{12}$$

Bayesian Analysis under the assumption of Maxwell prior

The posterior distribution is derived as;

$$p(r | x) = \frac{r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] - \frac{r^2}{a}} dr \tag{13}$$

Bayes estimate of the scale parameter using Maxwell prior under SELF;

$$E(r | x) = \int_0^\infty \frac{r^{n+3} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] - \frac{r^2}{a}} \times r dr \tag{14}$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{r}_{sl} = \int_0^\infty \frac{r^{n+4} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} dr - \left[\int_0^\infty \frac{r^{n+3} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} dr \right]^2 \quad (15)$$

Bayes estimate under PLF;

$$\sqrt{E(r^2 | x)} = \sqrt{\int_0^\infty \frac{r^{n+4} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} \times r^2 dr} \quad (16)$$

Posterior risk of the estimate under PLF;

$$Rr_{pl} = 2 \left[\sqrt{\int_0^\infty \frac{r^{n+4} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} dr} - \int_0^\infty \frac{r^{n+3} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+2} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} dr \right] \quad (17)$$

Bayesian analysis under the assumption of Chi-square prior

The posterior distribution is derived as follows;

$$p(r | x) = \frac{r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] + \frac{r^2}{2}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] + \frac{r^2}{2}} dr \quad (18)$$

Bayes estimate of the scale parameter using chi-square prior under SELF;

$$E(r | x) = \int_0^\infty \frac{r^{n+k} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] + \frac{r^2}{2}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] + \frac{r^2}{2}} dr \quad (19)$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{r}_{sl} = \int_0^\infty \frac{r^{n+k+1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} dr - \left[\int_0^\infty \frac{r^{n+k} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln S - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{S} \right)^r \right] - \frac{r^2}{a}} dr \right]^2 \quad (20)$$

Bayes estimate under PLF;

$$\sqrt{E(r^2 | x)} = \sqrt{\frac{\int_0^\infty r^{n+k+1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] + \frac{r^2}{2}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] + \frac{r^2}{2}} dr}} \quad (21)$$

Posterior risk under PLF;

$$Rr_{pl}^{\hat{}} = 2 \left[\frac{\int_0^\infty r^{n+k+1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] + \frac{r^2}{2}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] + \frac{r^2}{2}} dr} - \frac{\int_0^\infty r^{n+k} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] + \frac{r^2}{2}}{\int_0^\infty r^{n+k-1} \ell^{\sum_{i=1}^n \ln x_i^{r-1} - nr \ln s - 2 \sum_{i=1}^n \ln \left[1 + \left(\frac{x_i}{s} \right)^r \right] + \frac{r^2}{2}} dr} \right] \quad (22)$$

Simulation study

In this study, we perform a Monte-Carlo simulation study of 1000 runs using different sample sizes ($n = 10, 30, 50, 70$ & 100), the log-logistic samples were generated for the assumed values of scale ($r = 0.5$ & 4) and shape ($s = 1$ & 5) parameters. Also, for the purpose of comparison, we simulated the

estimates of the parameter under the assumption of non-informative priors (Uniform and Jeffrey's)

RESULTS AND DISCUSSION

We present the tables below showing the Bayes estimates and Bayes posterior risks of the scale parameter of log-logistic distribution under the assumption of Maxwell, Chi-square, Jeffrey's and Uniform priors.

Table 1.1: Bayes estimates and corresponding posterior risks (in parenthesis) under SELF assuming $r = 2$ and $s = 5$

prior		$r = 2, s = 5$				
n		10	30	50	70	100
UP		2.29837	2.05760	2.05760	2.03198	2.02652
		(0.37436)	(0.10230)	(0.05915)	(0.04115)	(0.02870)
JP		2.33704	2.09101	2.05648	2.03903	2.03144
		(0.38789)	(0.10184)	(0.05909)	(0.04157)	(0.02884)
MP	$a = 0.5$	2.29916	2.07581	2.07466	2.05394	2.02864
		(0.37057)	(0.10010)	(0.06023)	(0.04210)	(0.02876)
CP	$k = 0.5$	2.29815	2.09471	2.05657	2.04076	2.01757
		(0.36965)	(0.10217)	(0.05911)	(0.04159)	(0.02843)

Table 1.2: Bayes estimates and corresponding posterior risks (in parenthesis) under PLF assuming $r = 2$ and $S = 5$

prior		$r = 2, S = 5$				
n		10	30	50	70	100
UP		2.33422 (0.38762)	2.11060 (0.10353)	2.05935 (0.05931)	2.0427 (0.04162)	2.02818 (0.02875)
JP		2.22389 (0.14822)	2.08075 (0.04772)	2.03739 (0.02822)	2.02574 (0.02011)	2.02405 (0.01409)
MP	$a = 0.5$	1.78969 (0.09555)	1.91359 (0.04005)	1.95492 (0.02555)	1.96205 (0.01867)	1.96883 (0.01329)
CP	$k = 0.5$	1.76845 (0.09466)	1.93089 (0.04030)	1.95608 (0.02558)	1.96122 (0.01863)	1.98765 (0.01343)

Table 3.1: Bayes estimates and corresponding posterior risks (in parenthesis) under SELF assuming $r = 4$ and $S = 5$

Prior		$r = 4, S = 5$				
N		10	30	50	70	100
UP		4.65675 (1.52441)	4.17354 (0.40560)	4.13524 (0.23891)	4.07583 (0.16559)	4.04213 (0.11419)
JP		4.61245 (1.49157)	4.15120 (0.40049)	4.13438 (0.23946)	4.06779 (0.16511)	4.06450 (0.11542)
MP	$a = 0.5$	4.57950 (1.48794)	4.20030 (0.41170)	4.12213 (0.23726)	4.08276 (0.16666)	4.04214 (0.11417)
CP	$k = 0.5$	4.65022 (1.52836)	4.11897 (0.40827)	4.11665 (0.23644)	4.07928 (0.16622)	4.07505 (0.11605)

Table 3.1: Bayes estimates and corresponding posterior risks (in parenthesis) under SELF assuming $r = 4$ and $S = 5$

Prior		$r = 4, S = 5$				
N		10	30	50	70	100
UP		4.66903 (1.53658)	4.19074 (0.40957)	4.11138 (0.23662)	4.08929 (0.16696)	4.05861 (0.11512)
JP		4.41781 (0.29463)	4.10108 (0.09406)	4.06398 (0.05628)	4.07042 (0.04033)	4.03696 (0.02808)
MP	$a = 0.5$	2.50988 (0.12161)	3.17154 (0.06006)	3.44283 (0.04143)	3.58024 (0.03181)	3.68048 (0.02357)
CP	$k = 0.5$	2.50291 (0.12149)	3.19001 (0.06042)	3.44783 (0.04149)	3.57176 (0.03175)	3.68310 (0.02359)

From the simulation analysis, it can be seen that the Bayes estimates of the parameter approaches the true value as expected with increasing sample size. Furthermore, it is revealed that the estimates obtained under Maxwell and chi-square priors constitutes almost similar risks. However, it is obvious that estimates obtained under these informative priors outperforms those obtained under non-informative priors and again we can see that the PLF gives better estimates than the SELF

CONCLUSION

This study estimated the scale parameter of the log-logistic distribution using two informative priors (Maxwell and chi-square) under SELF and PLF. From the results obtained, it can be concluded that the PLF is preferable for obtaining the Bayes estimates of the scale parameter of the distribution. In addition, the performance of the informative priors used in this study is better than non-informative (Uniform and Jeffrey's) priors for each of the loss functions. therefore, the use of PLF under informative priors is recommended for the estimation of the parameter in question.

Further researches can be undertaken using other priors and loss functions.

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