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## MATLAB/SIMULINK ANALYSIS OF SECOND ORDER SYSTEM TRANSIENT RESPONSE

By

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### ABSTRACT

*The aim of this paper is to develop a systemic methodology for second order circuits by switched DC sources. The paper presents a unified approach through Matlab/Simulink to determining the transient response of linear RC, RL and RLC circuits; and although the methods presented in the chapter focus only on first and second order circuits, the approach to the transient solution is quite general. Practical applications of the second order circuits are presented and analogies are introduced to emphasize the general nature of the solution methods and their applicability to a wide range of physical systems.*

**Key Words:** RC, RL and RLC circuits, Second Order System, Overdamped System, Critically damped System, Underdamped System

### INTRODUCTION

Second-order state determined systems are described in terms of two state variables. Physical second-order system models contain two independent energy storage elements which exchange stored energy, and may contain additional dissipative elements; such models are often used to represent the exchange of energy between mass and stiffness elements in mechanical systems; between capacitors and inductors in electrical systems, and between fluid inertance and capacitance elements in hydraulic systems. In addition second-order system models are frequently used to represent the exchange of energy between two independent energy storage elements in different energy domains coupled through a two-port element, for example energy may be exchanged between a mechanical mass and a fluid capacitance (tank) through a piston, or between an electrical inductance and mechanical inertia as might occur in an electric motor. Engineers often use second-

order system models in the preliminary stages of design in order to establish the parameters of the energy storage and dissipation elements required to achieve a satisfactory response [1].

Second-order systems have responses that depend on the dissipative elements in the system. Some systems are oscillatory and are characterized by decaying, growing, or continuous oscillations. Other second order systems do not exhibit oscillations in their responses. In this section we define a pair of parameters that are commonly used to characterize second-order systems, and use them to define the conditions that generate non-oscillatory, decaying or continuous oscillatory, and growing (or unstable) responses.

In the following sections we transform the two state equations into a single differential equation in the output variable of interest, and then express this equation in a standard form.

**Transformation of State Equations to a Single Differential Equation**

The state equations  $\dot{x} = Ax + Bu$  for a linear second-order system with a single input are a pair of coupled first-order differential equations in the two state variables [2]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} * u \tag{1}$$

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1u$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2u \tag{2}$$

The state-space system representation may be transformed into a single differential equation in either of

the two state-variables. Taking the Laplace transform of the state equations [3].

$$\begin{aligned} (sI - A)X(s) &= BU(s) \\ X(s) &= (sI - A)^{-1}BU(s) \\ &= \frac{1}{\det[sI - A]} \begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} * U(s) \\ \det[sI - A]X(s) &= \begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} * U(s) \end{aligned}$$

from which

$$\frac{d^2x_1}{dt^2} - (a_{11} + a_{22})\frac{dx_1}{dt} + (a_{11}a_{22} - a_{12}a_{21})x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1) * u. \tag{3}$$

and

$$\frac{d^2x_2}{dt^2} - (a_{11} + a_{22})\frac{dx_2}{dt} + (a_{11}a_{22} - a_{12}a_{21})x_2 = b_2\frac{du}{dt} + (a_{12}b_1 - a_{22}b_2) * u. \tag{4}$$

which can be written in terms of the two parameters  $\zeta$  and  $\omega_n$

$$\frac{d^2x_1}{dt^2} + 2\zeta\omega_n\frac{dx_1}{dt} + \omega_n^2x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1) * u \tag{5}$$

$$\frac{d^2 x_2}{dt^2} + 2' \check{S}_n \frac{dx_2}{dt} + \check{S}_n^2 x_2 = b_2 \frac{du}{dt} + (a_{21}b_1 - a_{11}b_2) * u \tag{6}$$

where  $\check{S}_n$  is defined to be the *undamped natural frequency* with units of radians/second, and  $'$  is defined to be the system (dimensionless) *damping ratio*. These definitions may be compared to Eqs. (3) and (4), to give the following relationships:

$$\check{S}_n = \sqrt{a_{11}a_{22} - a_{12}a_{21}} \tag{7}$$

$$\begin{aligned} ' &= -\frac{1}{2\check{S}_n}(a_{11} + a_{22}) \\ &= \frac{-(a_{11} + a_{22})}{2\sqrt{a_{11}a_{22} - a_{12}a_{21}}}. \end{aligned} \tag{8}$$

The undamped natural frequency and damping ratio play important roles in defining second-order system responses, similar to the role of the time constant in first-

order systems, since they completely define the system homogeneous equation [4].

**Homogeneous Second-Order Equation**

For any system variable  $y(t)$  in a second-order system, the homogeneous equation is found by setting the input  $u(t) \equiv 0$ .

$$\frac{d^2 y}{dt^2} + 2\check{S}_n ' \frac{dy}{dt} + \check{S}_n^2 y = 0. \tag{9}$$

The solution,  $y_h(t)$ , to the homogeneous equation is found by assuming the general exponential form

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{10}$$

where  $C_1$  and  $C_2$  are constants defined by the initial conditions, and the eigenvalues  $\lambda_1$  and  $\lambda_2$  are the roots of the characteristic equation

$$\det[sI - A] = s^2 + 2' \check{S}_n s + \check{S}_n^2 = 0 \tag{11}$$

found using the quadratic formula:

$$\lambda_1, \lambda_2 = -' \check{S}_n \pm \check{S}_n \sqrt{'^2 - 1}. \tag{12}$$

If  $\lambda_1 = \lambda_2 = \lambda$ , the two roots are equal ( $\lambda_1 = \lambda_2 = \lambda$ ), a modified form for the homogeneous solution is necessary:

$$y_c(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} \tag{13}$$

In either case the homogeneous solution consists of two independent exponential components, with two arbitrary constants,  $C_1$  and  $C_2$ , whose values are selected to make the solution satisfy a given pair of initial conditions. In general the value of the output  $y(0)$  and its derivative  $\dot{y}(0)$  at time  $t = 0$  are used to provide the

necessary information. The initial conditions for the output variable may be specified directly as part of the problem statement, or they may have to be determined from knowledge of the state variables  $x_1(0)$  and  $x_2(0)$  at time  $t = 0$ . The homogeneous output equation may be used to compute  $y(0)$  directly from elements of the A and C matrices [5]

$$y(0) = c_1 x_1(0) + c_2 x_2(0) \tag{14}$$

and the value of the derivative  $\dot{y}(0)$  may be determined by differentiating the output equation and substituting for

the derivatives of the state variables from the state equations:

$$\begin{aligned} \dot{y}(0) &= c_1 \dot{x}_1(0) + c_2 \dot{x}_2(0) \\ &= c_1 (a_{11} x_1(0) + a_{12} x_2(0)) + c_2 (a_{21} x_1(0) + a_{22} x_2(0)) \end{aligned} \tag{15}$$

To illustrate the influence of damping ratio and natural frequency on the system response, we consider the response of an unforced system output variable with initial output conditions of

$y(0) = y_0$ , and  $\dot{y}(0) = 0$ . If the roots of the characteristic equation are distinct, imposing these initial conditions on the general solution of Eq. (10) gives:

$$\begin{aligned} y(0) &= y_0 = C_1 + C_2 \\ \left. \frac{dy}{dt} \right|_{t=0} &= 0 = \lambda_1 C_1 + \lambda_2 C_2. \end{aligned} \tag{16}$$

With the result that:

$$C_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} y_0 \quad \text{and} \quad C_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} y_0 \tag{17}$$

For this set of initial conditions the homogeneous solution is therefore

$$y_h(t) = y_0 \left[ \left( \frac{\lambda_2}{\lambda_2 - \lambda_1} \right) * e^{\lambda_1 t} + \left( \frac{\lambda_1}{\lambda_1 - \lambda_2} \right) * e^{\lambda_2 t} \right] \tag{18}$$

$$= y_0 \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ \frac{1}{\lambda_1} e^{\lambda_1 t} - \frac{1}{\lambda_2} e^{\lambda_2 t} \right]. \tag{19}$$

If the roots of the characteristic equation are identical  $\lambda_1 = \lambda_2 = \lambda$ , the solution is based on Eq. (13) and is:

$$y_h(t) = y_0 [e^{\lambda t} - \lambda t e^{\lambda t}] \tag{20}$$

The system response depends directly on the values of the damping ratio  $\zeta$  and the undamped natural frequency  $\omega_n$ . Four separate cases are described below [6]

**Overdamped System ( $\zeta > 1$ ):** When the damping ratio  $\zeta$  is greater than one, the two roots of the characteristic equation are real and negative:

$$\lambda_1, \lambda_2 = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \tag{21}$$

From Eq. (19) the homogeneous response is

$$y_h(t) = y_0 \left[ \frac{-\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} - \frac{-\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \right] \tag{22}$$

Which is the sum of two decaying real exponentials, each with a different decay rate that defines a time constant

$$\tau_1 = -\frac{1}{\lambda_1}, \quad \tau_2 = -\frac{1}{\lambda_2}. \tag{23}$$

The response exhibits no overshoot or oscillation, and is known as an *overdamped* response. Figure 1 shows this response as a function of  $\zeta$  using a normalized time scale of  $\omega_n t$ .

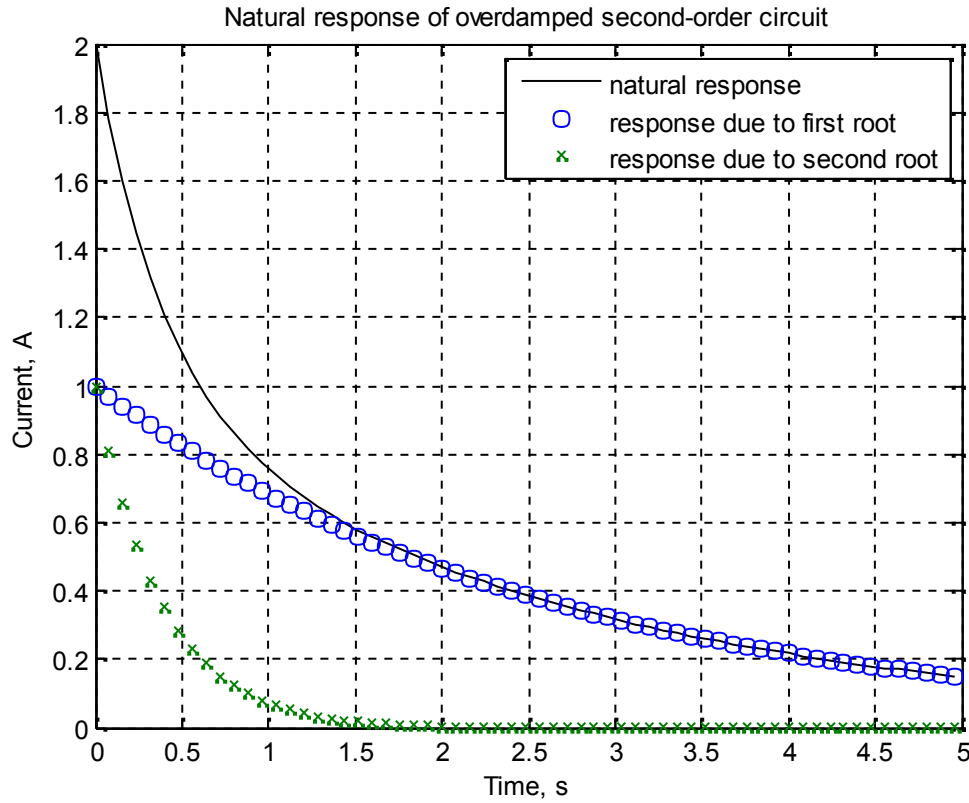


Figure 1: Normalized initial condition response of an overdamped second-order system as a function of the damping ratio  $\zeta$  ( $r_1 = r_2; \zeta = 1.5; \check{S}_n = 1$ )

**Critically Damped System ( $\zeta = 1$ ):** When the damping ratio  $\zeta = 1$  the roots of the characteristic equation are real and identical [7].

$$s_1 = s_2 = -\check{S}_n \tag{24}$$

The solution to the initial condition response is found from Eq. (56):

$$y_h(t) = y_0 \left[ e^{-\check{S}_n t} + \check{S}_n t e^{-\check{S}_n t} \right] \tag{25}$$

which is shown in Figure 2. This response form is known as a *critically damped* response because it marks the transition between the non-oscillatory overdamped response and the oscillatory response.

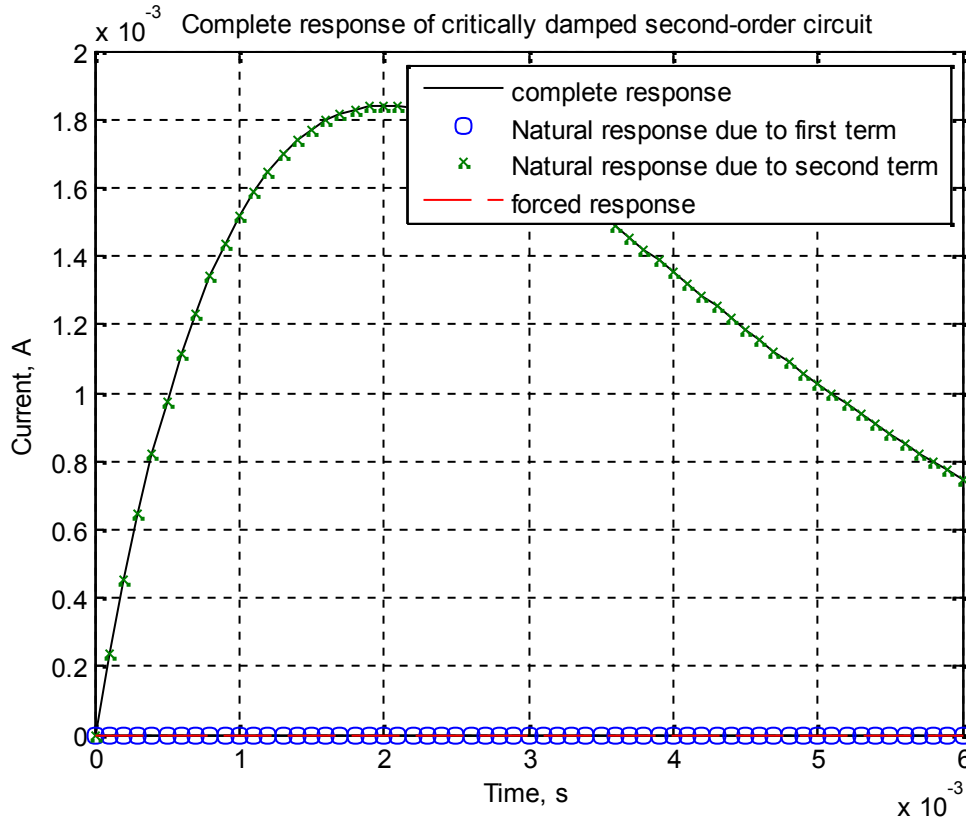


Figure 2: Normalized initial condition response of a critically damped second-order system as a function of the damping ratio  $\zeta$  ( $r_1 = r_2; \zeta = 1.5; \tilde{S}_n = 1$ )

**Underdamped System ( $0 < \zeta < 1$ ):** When the damping ratio is greater than or equal to zero but less than 1, the two roots of the characteristic equation are complex conjugates with negative real parts [7]:

$$s_1, s_2 = -\zeta \tilde{S}_n \pm j \tilde{S}_n \sqrt{1 - \zeta^2} = -\zeta \tilde{S}_n \pm j \tilde{S}_d \tag{26}$$

where  $j = \sqrt{-1}$ , and where  $\tilde{S}_d$  is defined to be the *damped natural frequency*:

$$\tilde{S}_d = \tilde{S}_n \sqrt{1 - \zeta^2} \tag{27}$$

The response may be determined by substituting the values of the roots in Eq. (26) into Eq. (19):

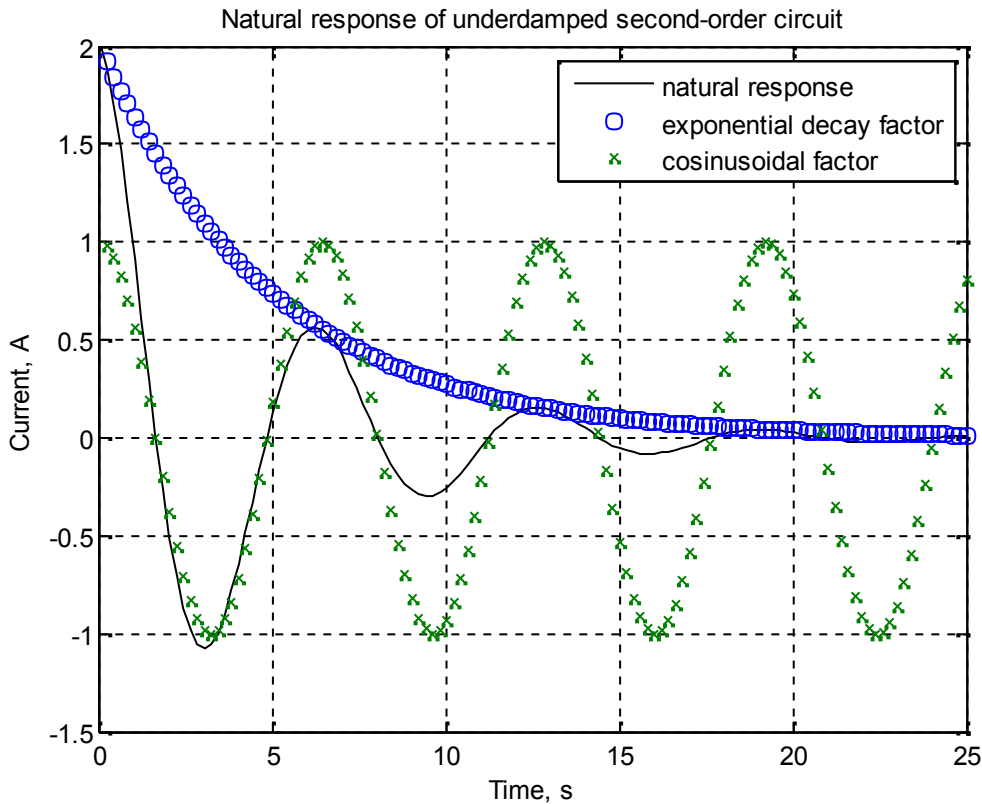
$$\begin{aligned}
 y_h(t) &= y_0 \left[ \left( \frac{-\zeta \check{S}_n - j\check{S}_d}{-2j\check{S}_d} \right) e^{(-\zeta \check{S}_n + j\check{S}_d)^*t} + \left( \frac{-\zeta \check{S}_n - j\check{S}_d}{-2j\check{S}_d} \right) e^{(-\zeta \check{S}_n - j\check{S}_d)^*t} \right] \\
 &= y_0 e^{-\zeta \check{S}_n t} \left[ \frac{e^{+j\check{S}_d t} + e^{-j\check{S}_d t}}{2} + \left( \frac{\zeta \check{S}_n}{\check{S}_d} \right) \frac{e^{+j\check{S}_d t} - e^{-j\check{S}_d t}}{2j} \right]
 \end{aligned}
 \tag{28}$$

When the Euler identities  $\cos r = (e^{+jr} + e^{-jr})/2$  and  $\sin r = (e^{+jr} - e^{-jr})/2j$  are substituted the solution is:

$$\begin{aligned}
 y_h(t) &= y_0 e^{-\zeta \check{S}_n t} \left[ \cos \check{S}_d t + \frac{\zeta \check{S}_n}{\check{S}_d} \sin \check{S}_d t \right] \\
 &= y_0 \frac{e^{-\zeta \check{S}_n t}}{\sqrt{1-\zeta^2}} * \cos(\check{S}_d t - \{)
 \end{aligned}
 \tag{29}$$

Where the phase angle  $\{$  is

$$\{ = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}
 \tag{30}$$



**Figure 3:** Normalized initial condition response of an underdamped second-order system as a function of the damping ratio  $\zeta$  ( $\alpha_1 = \alpha_2 = 1; \xi = 0.2; \omega_n = 1$ )



The initial condition response for an *underdamped* system is a damped cosine function, oscillating at the damped natural frequency  $\tilde{\omega}_d$  with a phase shift  $\phi$ , and with the rate of decay determined by the exponential term  $e^{-\tilde{\omega}_d t}$ . The response for underdamped second-order systems are plotted against

$$y_h(t) = y_0 * \cos(\tilde{\omega}_d t) \tag{31}$$

and persists for all time. (The term "undamped natural frequency" for  $\tilde{\omega}_n$  is derived from this situation, because a system with  $\zeta = 0$  oscillates at a frequency of  $\tilde{\omega}_n$ .) As the damping ratio increases from zero, the frequency of oscillation  $\tilde{\omega}_d$  decreases, as shown by Eq. (27), until at a damping ratio of unity, the value of  $\tilde{\omega}_d = 0$  and the response consists of a sum of real decaying exponentials.

$$\begin{aligned}
 DR &= \frac{y(t + T_p)}{y(t)} \quad \text{provided} \quad y(t) \neq 0 \\
 &= \frac{e^{-\zeta \tilde{\omega}_n (t + 2\pi / \tilde{\omega}_d)}}{e^{-\zeta \tilde{\omega}_n t}} \\
 &= e^{-2\pi \zeta / \sqrt{1 - \zeta^2}}
 \end{aligned} \tag{32}$$

The decay ratio is unity if the damping ratio is zero, and decreases as the damping ratio increases, reaching a value of zero as the damping ratio approaches unity.

**CONCLUSION**

The response can be classified as one of three types of damping that describes the output in relation to the steady-state response. An underdamped response is one that oscillates within a decaying envelope. The more underdamped the system, the more oscillations and

normalized time  $\tilde{\omega}_n t$  for several values of damping ratio in Figure 3.

For damping ratios near unity, the response decays rapidly with few oscillations, but as the damping is decreased, and approaches zero, the response becomes increasingly oscillatory. When the damping is zero, the response becomes a pure oscillation

The decay rate of the amplitude of oscillation is determined by the exponential term  $e^{-\zeta \tilde{\omega}_n t}$ . It is sometimes important to determine the ratio of the oscillation amplitude from one cycle to the next. The cosine function is periodic and repeats with a period  $T_p = 2\pi / \tilde{\omega}_d$ , so that if the response at an arbitrary time  $t$  is compared with the response at time  $t + T_p$ , an amplitude decay ratio DR may be defined as:

longer it takes to reach steady-state. Here damping ratio is always < 1.

A critically damped response is the response that reaches the steady-state value the fastest without being underdamped. It is related to critical points in the sense that it straddles the boundary of underdamped and overdamped responses. Here, damping ratio is always equal to one. There should be no oscillation about the steady state value in the ideal case. An overdamped response is the response that does not oscillate about the steady-state value but takes longer to reach than the

critically damped case. Here damping ratio is  $>1$  it is the response of a system with respect to the input as a function of time

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