

ON GENERALIZED WEIBULL-RAYLEIGH DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS

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ABSTRACT:

Rayleigh distribution as one of the widely used continuous probability distribution was discovered in 1980 by a researcher called Rayleigh. He discovered it from the amplitude of sound resulting from different important sources. The Rayleigh distribution enjoys a wide range of applications comprising life testing experiments, reliability analysis, applied statistics and clinical studies. The distribution is a special case of the two parameter Weibull distribution when the shape parameter takes on a value 2. In this article, a Generalized Weibull-Rayleigh Distribution (GWRD) is presented. The respective definitions of its probability density function (pdf) and cumulative distribution function (cdf) using the generalized family proposed by earlier research were provided. Some properties of the new distribution such as moments, moment generating function, characteristics function, quantile function, survival function, hazard function and order statistics were studied. The estimation of the distribution's parameters was conducted using the method of maximum likelihood. The performance of the proposed probability distribution (GWRD) was compared to some other generalizations of Rayleigh distribution using three lifetime datasets. The results obtained indicated that GWRD outperforms other similar distributions comprising Weibull-Rayleigh, Transmuted Rayleigh and the conventional Rayleigh distributions.

Keywords: Rayleigh Distribution, Reliability analysis, Maximum likelihood estimation, order statistics, Generalized Weibull-Rayleigh distribution, parameters, application.

INTRODUCTION

The Rayleigh distribution has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. The distribution is a special case of the two parameter

$$g(x ; \theta) = \theta x e^{-\frac{\theta}{2}x^2} \tag{1}$$

And the corresponding cumulative distribution function (cdf) is given as

$$G(x ; \theta) = 1 - e^{-\frac{\theta}{2}x^2} \tag{2}$$

for $x > 0, \theta > 0$ where θ is the scale parameter.

There exist so many generalized families of distributions proposed by different researchers that are used in extending so many standard or classical

Weibull distribution with the shape parameter equal to 2. A random variable X is said to have a Rayleigh distribution with parameter θ if its probability density function (pdf) is given by:

distributions to produce compound distributions found to be better than the classical ones. Some generalizations of the Rayleigh distribution led to the development of

other compound distributions such as the generalized Rayleigh distribution by Kundu *et al.*, (2005), Bivariate generalized Rayleigh distribution by Abdel-Hady (2013), Transmuted Rayleigh distribution by Merovci (2013) and the Transmuted Inverse Rayleigh distribution by Ahmad *et al.*, (2014). In a similar manner, for any continuous

$$f(x) = r s \frac{g(x)}{G'(x)} \left(-\log[1 - G(x)]\right)^{s-1} e^{-r(-\log[1-G(x)])^s} \tag{3}$$

and

$$F(x) = 1 - e^{-c(-\log[1-G(x)])^s} \tag{4}$$

Where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution respectively while $G'(x) = 1 - G(x)$ and $\alpha > 0$ and $\beta > 0$ are the scale and shape parameters respectively.

The aim of this paper is to introduce a new continuous distribution called the Generalized Weibull-Rayleigh distribution (*GWRD*) from the proposed family by Cordeiro *et al.* (2015). The remaining parts of this paper are presented in sections as follows: We defined the new distribution and give its plots in section 2.

$$F(x) = 1 - e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]\right)^s} \tag{5}$$

$$f(x) = \frac{r s x e^{-\frac{r}{2}x^2}}{\left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]} \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]\right)^{s-1} e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]\right)^s} \tag{6}$$

The *pdf* and *cdf* of the *GWRD* at chosen parameter values are displayed in **figures 1** and **2** as below.

distribution with *cdf* $G(x)$, and *pdf* $g(x)$. Cordeiro *et al.* (2015) defined the generalized Weibull family of distributions (*GW-B*) with two extra parameters $\alpha > 0$ and $\beta > 0$ to have its *pdf* $f(x)$ and *cdf* $F(x)$ respectively given by:

Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. In section 5, we carryout application of the proposed model with others to three real life data sets. Lastly, in section 6, we make some useful conclusions.

The *GWRD*

By considering the *pdf* (1) and *cdf* (2) of the Rayleigh distribution. The *cdf* and *pdf* of the *GWRD* are obtained from equation (3) and (4) as:

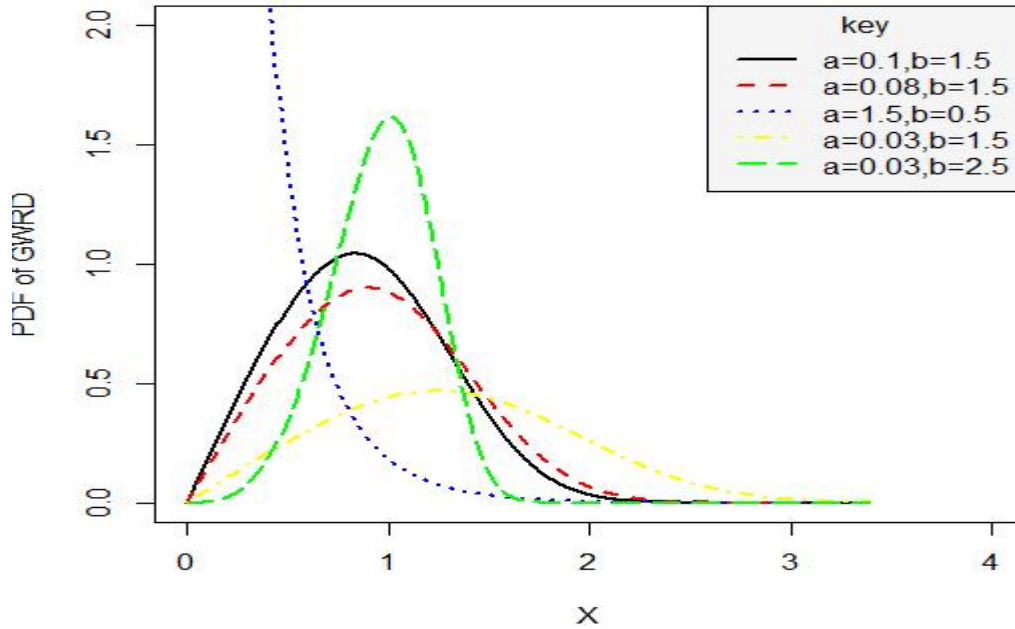


Figure 1: The pdf plot of the GWRD using different parameter values where $a = r, b = s$ for $n = 6.5$.

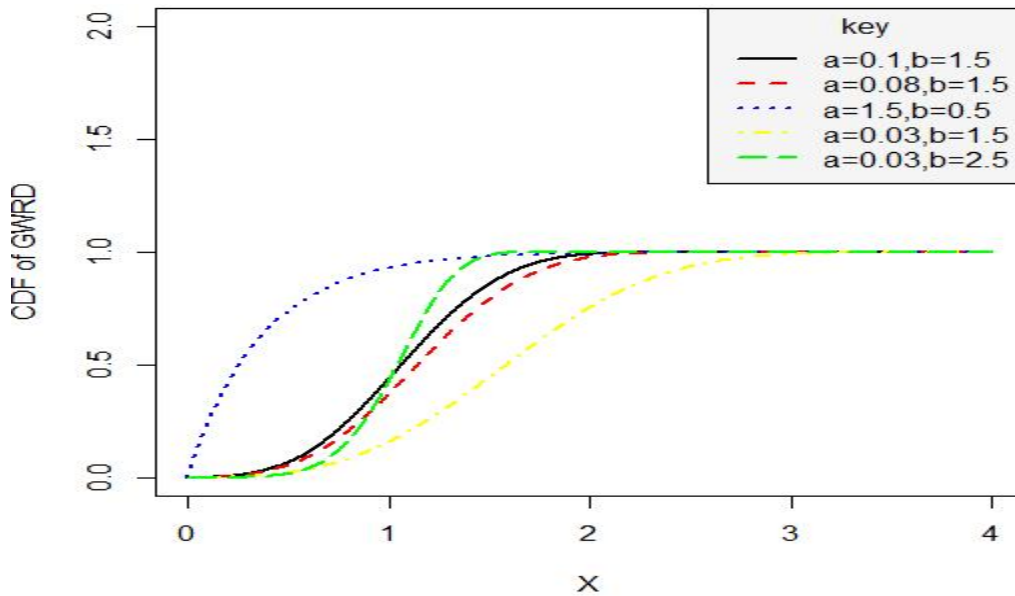


Figure 2: The cdf plot of the GWRD at different parameter values where $a = r, b = s$ for $n = 6.5$. The plot for the pdf shows that the GWRD is positively skewed and hence good model for modeling positively-skewed datasets.

PROPERTIES

In this section, we defined and discuss some properties of the GWRD distribution.

Moments

Let X denote a continuous random variable, the r^{th} moment of X is given by;

$$\tilde{\sim}'_n = E(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx = \int_0^{\infty} x^n f(x) dx \tag{7}$$

where $f(x)$ the *pdf* of the Generalized Weibull-Rayleigh distribution is as given in equation (6).

Hence,

$$\tilde{\sim}'_n = E(X^n) = \left(\frac{1}{r}\right)^{\frac{n}{2s}} \left(\frac{2}{r}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2s} + 1\right) \tag{8}$$

The Mean

The mean of the *GWRD* can be obtained from the n^{th} moment of the distribution when $n=1$ as follows:

$$E(X) = \left(\frac{1}{r}\right)^{\frac{1}{2s}} \left(\frac{2}{r}\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{2s} + 1\right) \tag{9}$$

Also the second moment of the *GWRD* is obtained from the n^{th} moment of the distribution when $n=2$ as

$$E(X^2) = \left(\frac{1}{r}\right)^{\frac{2}{s}} \left(\frac{2}{r}\right) \Gamma\left(\frac{2}{s} + 1\right) \tag{10}$$

The Variance

The n^{th} central moment or moment about the mean of X , say μ_n , can be obtained as

$$\mu_n = E(X - \tilde{\sim}'_1)^n = \sum_{i=0}^n (-1)^i \binom{n}{i} \tilde{\sim}'_1^i \tilde{\sim}'_{n-i} \tag{11}$$

The variance of X for *GWRD* is obtained from the central moment when $n=2$, that is,

$$Var(X) = E(X^2) - \{E(X)\}^2 \tag{12}$$

$$Var(X) = \left(\frac{1}{r}\right)^{\frac{2}{s}} \left(\frac{2}{r}\right) \left\{ \Gamma\left(\frac{2}{s} + 1\right) - \left[\Gamma\left(\frac{1}{2s} + 1\right) \right]^2 \right\} \tag{13}$$

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

Moment Generating Function

The *mgf* of a random variable X can be obtained by:

As the name implies, the moment generating function (*mgf*) is used to generate the moments of X by differentiation i.e., for any real t

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} f(x) dx \tag{14}$$

$$M_x(t) = E(e^{tx}) = \frac{rs_n^s}{2^{s-1}} \int_0^{\infty} x^{2s-1} e^{tx} e^{-r\left(\frac{x}{2}\right)^s} dx = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tilde{\sim}'_n \tag{15}$$

where $\tilde{\sim}'_n$ is as defined in equation (8).

Characteristics Function

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions

and in analysis of linear combination of independent random variables.

The characteristics function of a random variable X is given by;

$$\phi_x(t) = E(e^{itx}) = E[\cos(tx) + i \sin(tx)] = E[\cos(tx)] + E[i \sin(tx)] \tag{16}$$

Simple algebra and power series expansion proves that

$$\phi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \sim_{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \sim_{2n+1} \tag{17}$$

Where μ'_{2n} and μ'_{2n+1} are the moments of X for $n=2n$ and $n=2n+1$ respectively and can be obtained from μ'_n as

$$\sim_{2n} = E(X^{2n}) = \left(\frac{1}{r}\right)^{\frac{n}{s}} \left(\frac{2}{r}\right)^n \Gamma\left(\frac{n}{s} + 1\right)$$

and

$$\sim_{2n+1} = E(X^{2n+1}) = \left(\frac{1}{r}\right)^{\frac{2n+1}{2s}} \left(\frac{2}{r}\right)^{\frac{2n+1}{2}} \Gamma\left(\frac{2n+1}{2s} + 1\right)$$

Quantile function for the GWRD.

The quantile function, say $X=Q(u)$, of the *GWRD* can be obtained as the inverse of Equation (5) as:

$$F(x) = 1 - e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s}$$

$$X = Q(u) = \sqrt{\frac{2}{r} \left(\frac{1}{r} \ln \left(\frac{1}{1-u} \right) \right)^{\frac{1}{s}}} \tag{18}$$

By using (18) above, the median of X from the *GWRD* is simply obtained by setting $u = 0.5$ while random numbers can be generated from *GWRD* by setting $X = Q(u)$, where u is a uniform variate on the unit interval $(0,1)$.

Skewness and kurtosis

skewness and kurtosis's finite coefficients are calculated from (18) using the Bowley's Skewness (Kenney & Keeping, 1962) and Moor's kurtosis depend on quartile with the following statistical expressions; Bowley's skewness is defined as:

$$B = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \tag{19}$$

And the Moor's Kurtosis (Moors, 1998) based on octiles is given by:

$$M = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})} \tag{20}$$

where $Q(\cdot)$ is any quartile or octile of interest.

Reliability analysis of the GWRD.

There are two significant functions in reliability analysis, the survival and hazard functions. The survival

function is a function of success or continuity, it tells us the probability that a variable will survive beyond a specified time. This function is given by:

$$S(X) = P(X > x) = 1 - F(x) \tag{21}$$

Applying the GWRD in (17), the survival function for the GWRD is obtained as:

$$S(x) = e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2}x^2} \right) \right] \right)^s} \tag{22}$$

For $x > 0$, where α, β, μ and σ are the parameters.

A plot for the survival function of the GWRD using different parameter values is as shown in Figure 3 below:

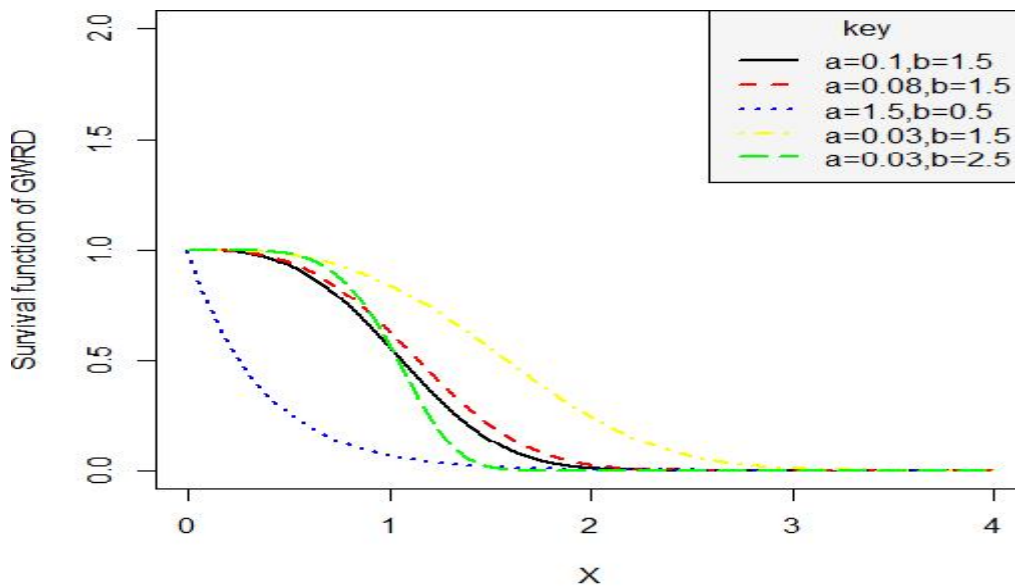


Figure. 3: Survival function of the GWRD at different parameter values where $a = r, b = s$ for $r = 6.5$.

Figure 3 tells us that the chances of success are higher at initial time and decreases as X increases and maintains zero (0) level as X tends to plus infinity.

The hazard or failure function is obtained mathematically as the ratio of the pdf, $f(x)$ to the survival function $S(X)$. It is given by:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1-F(x)} \tag{23}$$

Meanwhile, the expression for the hazard rate of the *GWWD* is given by

$$h(x) = rs_n x \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2}x^2} \right) \right] \right)^{s-1} \tag{24}$$

where $r, s, t > 0$ and $-\infty \leq x \leq \infty$.

These are some possible curves for the hazard rate at chosen values of the model parameters

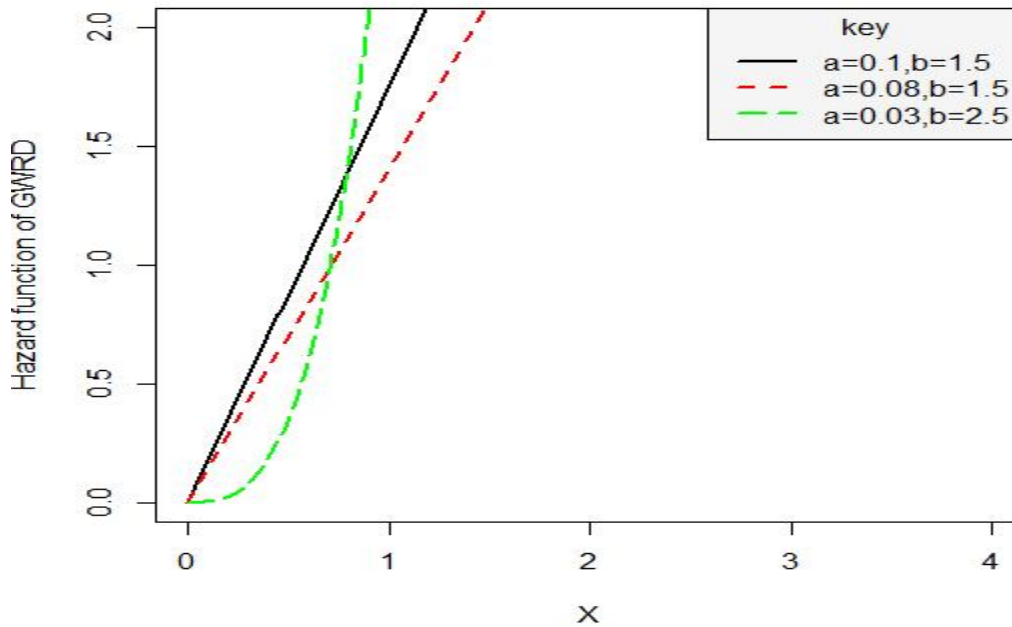


Figure 4: Hazard function of the *GWWD* at different parameter values where $a = r, b = s$ for $n = 6.5$.

It is clear from figure 4 that the failure for any variable following the *GWWD* increases as X increases. It becomes uncontrollably higher as the value of X gets larger. This connotes that the *GWWD* will be used for modeling components with increasing risk, that is, where risk or hazard increases with time.

Order Statistics

In this section, we derive closed form expressions for the *pdf* of the a^{th} order statistics of the

GWWD. If X_1, X_2, \dots, X_n is a random sample from the *GWWD* and also let $X_{1:n}, X_{2:n}, \dots, X_{i:n}$ be the corresponding order statistic obtained from this sample. The *pdf*, $f_{a:n}(x)$ of the a^{th} order statistic can be obtained by:

$$f_{a:n}(x) = \frac{n!}{(a-1)!(n-a)!} \sum_{k=0}^{n-a} (-1)^k \binom{n-a}{k} f(x) F(x)^{k+a-1} \tag{25}$$

where $f(x)$ and $F(x)$ are the *pdf* and *cdf* of the proposed distribution respectively. Using (5) and (6), the *pdf* of the a^{th} order statistics $X_{a:n}$, can be expressed from (25) as;

$$f_{a:n}(x) = \frac{n!}{(a-1)!(n-a)!} \sum_{k=0}^{n-a} (-1)^k \binom{n-a}{k} \times \left[r s_{\alpha} x \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^{s-1} e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s} \right] \times \left[1 - e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s} \right]^{a+k-1} \tag{26}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *GWGD* are given by;

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \times \left[r s_{\alpha} x \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^{s-1} e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s} \right] \times \left[1 - e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s} \right]^k \tag{27}$$

and

$$f_{n:n}(x) = n \left[r s_{\alpha} x \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^{s-1} e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s} \right] \times \left[1 - e^{-r \left(-\log \left[1 - \left(1 - e^{-\frac{r}{2} x^2} \right) \right] \right)^s} \right]^{n-1} \tag{28}$$

ESTIMATION OF PARAMETER

Let X_1, \dots, X_n be a sample of size n containing independently and identically distributed

random variables drawn from the *GWGD* with unknown parameters α , β , and ϑ . The *pdf* of the *GWGD* is then given as:

$$f(x) = \frac{rs_n x e^{-\frac{r}{2}x^2}}{\left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]} \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]\right)^{s-1} e^{-r\left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x^2}\right)\right]\right)^s} \tag{29}$$

The likelihood function is given as:

$$L(X_1, X_2, \dots, X_n / r, s, n) = \frac{(rs_n)^n \prod_{i=1}^n x_i}{\prod_{i=1}^n \left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]} e^{-\frac{r}{2} \sum_{i=1}^n x_i^2} \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right)^{s-1} \times e^{\sum_{i=1}^n \left\{-r\left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right)^s\right\}} \tag{30}$$

The log-likelihood function, $l(\xi) = \log L(X_1, X_2, \dots, X_n / r, s, n)$ is obtained by:

$$l(\xi) = n \log r + n \log s + n \log n + \sum_{i=1}^n \log x_i - \frac{n}{2} \sum_{i=1}^n x_i^2 - \sum_{i=1}^n \log \left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right] + (s-1) \sum_{i=1}^n \log \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right) - r \sum_{i=1}^n \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right)^s \tag{31}$$

Taking the partial derivative of the log likelihood function, $l(\varphi)$ with respect to α , β and ϑ respectively, we get the *MLEs* as follows:

$$\frac{\partial l(\xi)}{\partial r} = \frac{n}{r} - \sum_{i=1}^n \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right)^s \tag{32}$$

$$\frac{\partial l(\xi)}{\partial s} = \frac{n}{s} + \sum_{i=1}^n \log \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right) - r \sum_{i=1}^n \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right)^s \log \left(-\log\left[1 - \left(1 - e^{-\frac{r}{2}x_i^2}\right)\right]\right) \tag{33}$$

$$\frac{\partial l(\xi)}{\partial n} = \frac{n}{n} + \frac{(s-1)}{2} \sum_{i=1}^n \frac{x_i^2 e^{-\frac{r}{2}x^2}}{\left(-\log\left[1-\left(1-e^{-\frac{r}{2}x^2}\right)\right]\right)\left(1-\left(1-e^{-\frac{r}{2}x^2}\right)\right)} - \frac{rs}{2} \sum_{i=1}^n \frac{\left(-\log\left[1-\left(1-e^{-\frac{r}{2}x^2}\right)\right]\right)^{s-1} \left(x_i^2 e^{-\frac{r}{2}x^2}\right)}{\left[1-\left(1-e^{-\frac{r}{2}x^2}\right)\right]} \tag{34}$$

Solving the equations of $\frac{dl}{dr} = 0$, $\frac{\partial l}{\partial s} = 0$, and $\frac{\partial l}{\partial n} = 0$ for each parameter shall provide the maximum likelihood estimates (MLEs) of parameters r, s and n . However, these solutions cannot be obtained above analytically except numerically with the help of some statistical software/packages with data sets are available.

APPLICATIONS

In this section, we have considered the adequacy of the *GWRD* compared to those of three generalizations of the Rayleigh model including the Weibull-Rayleigh distribution (*WRD*), the Transmuted Rayleigh distribution (*TRD*) and the Rayleigh distribution (*RD*) and three real life data sets for fitting the above selected models with their descriptive statistics. The

summary of these data sets is also provided in Tables 1, 2 and 3 as follows:

Dataset I: This data set represents the strength of 1.5cm glass fibers initially collected by members of staff at the UK national laboratory. It has been used by Afify and Aryal (2016), Barreto-Souza *et al.* (2011), Bourguignon *et al.* (2014), Oguntunde *et al.* (2015) as well as Smith and Naylor (1987). Its summary is given as follows:

Table 1: Summary Statistics for Dataset I

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

Dataset II: This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients

receiving an analgesic and reported by Gross *et al.* (1975) and has been used by shanker *et al.* (2016).

Table 2: Summary Statistics for Data set II

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.8625	7.1854

Dataset III: The third dataset represents 66 observations of the breaking stress of carbon fibres of 50mm length (in GPa) given by Nicholas and Padgett

(2006). The descriptive statistics for this data are as follows:

Table 3: Descriptive statistics for Dataset III

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	66	0.390	2.178	2.835	3.278	2.760	4.900	0.795	-0.1285	3.2230

We also provide some histograms and densities for the three data sets as shown in **Figures 5, 6** and **7**.

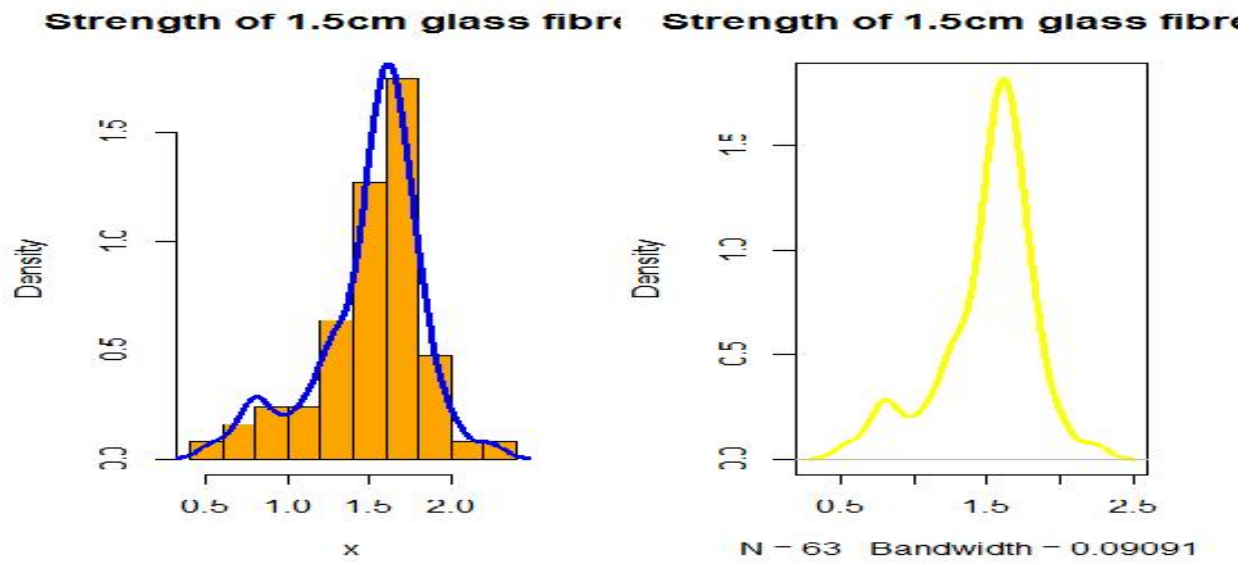


Figure 3: A histogram and density plot for the strength of 1.5cm glass fibres (Data set I)

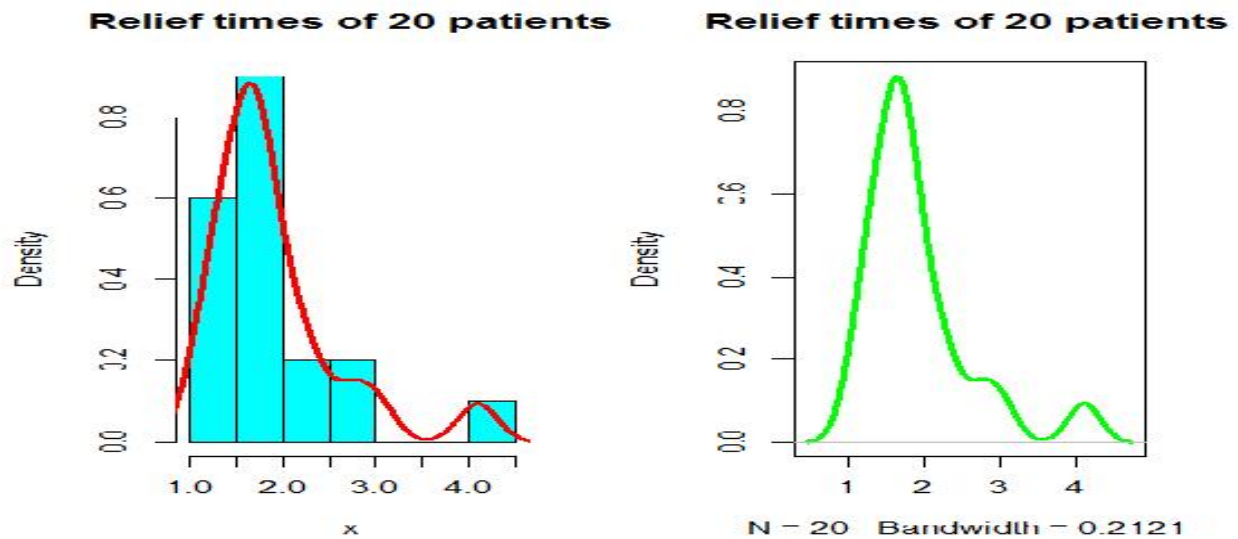


Figure 4: A Histogram and density plot for the Relief times of 20 patients (Data set II)

Breaking stress of carbon fib Breaking stress of carbon fib

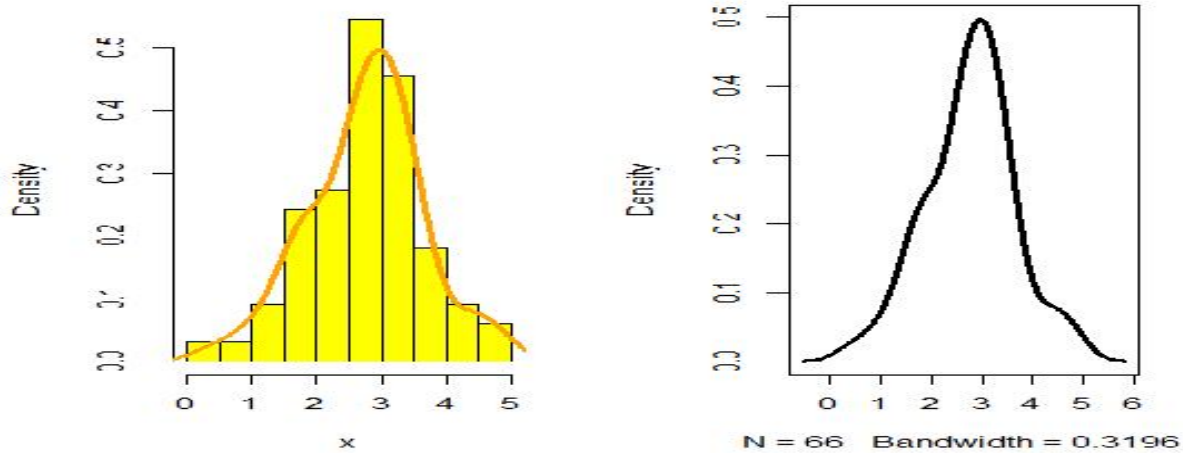


Figure 5: A Histogram and density plot for the Breaking stress of carbon fibres (Data set III)

From the descriptive statistics in tables 1, 2 and 3 as well as the histograms and densities shown above in figures 5, 6 and 7 for the three data sets respectively, we observed that the first data set is negatively skewed, the second data set is positively skewed while the third data is approximately normal and therefore suitable for distributions that are skewed to the left, right and symmetry respectively. Besides the models listed above, these three data sets could also be analyzed using a distribution which has various shapes

depending on the values of the parameters and which the generalized Weibull-Rayleigh distribution is a special case. For us to assess the models listed above, we made use of some criteria: the *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *HQIC* (Hannan Quin information criterion).

Decision benchmark: The model with the lowest values of these statistics would be chosen as the best model to fit the data.

Table 4: Performance of the selected models using the *AIC*, *CAIC*, *BIC* and *HQIC* values of the models evaluated at the *MLEs* based on data set I.

Distributions	Parameter estimates	$-l(-\log\text{-likelihood value})$	<i>AIC</i>	<i>CAIC</i>	<i>BIC</i>	<i>HQIC</i>	Ranks of models performance
<i>GWRD</i>	$\hat{\theta} = 0.3965$ $\hat{\alpha} = 3.6027$ $\hat{\beta} = 8.9e+6$	-890.6894	-1775.3788	-1775.2826	-1775.9808	-1779.8481	1
<i>WRD</i>	$\hat{\theta} = 0.3918$ $\hat{\alpha} = 2.2981$ $\hat{\beta} = 2.2634$	14.6575	35.3150	35.7218	34.7130	30.8457	2
<i>TRD</i>	$\hat{\theta} = 1.5061$ $\hat{\alpha} = -1.6251$ $\hat{\beta} = 2.2634$	27.0675	58.1350	58.3350	57.7337	55.1555	3
<i>RD</i>	$\hat{\theta} = 0.8426$	49.7909	101.5818	101.6474	101.3811	100.0920	4

In **Table 4**, the values of the parameter MLEs and the corresponding values of $-ll$, AIC , BIC , $CAIC$ and $HQIC$ for each model show that the $GWRD$ has better performance compared to the WRD , TRD and RD . This

also agrees with the fact that generalizing any continuous distribution provides a distribution with a better fit than the classical distribution.

Table 5: Performance of the distribution using the AIC , $CAIC$, BIC and $HQIC$ values of the models based on data set II.

Distributions	Parameter estimates	$-ll$ ($-\log$ -likelihood value)	AIC	$CAIC$	BIC	$HQIC$	Ranks of models performance
$GWRD$	$\hat{\theta}=0.1166$ $\hat{\alpha}=1.0563$ $\hat{\beta}=1.8e+7$	-290.5456	-575.0912	-573.5912	-577.1881	-580.4055	2
WRD	$\hat{\theta}=0.0264$ $\hat{\alpha}=38.4261$ $\hat{\beta}=1.3273$	20.9109	47.8218	49.3218	45.7249	42.5075	3
TRD	$\hat{\theta}=1.3273$ $\hat{\alpha}=1.3652$ $\hat{\lambda}=-1.2e+26$	-1154.409	-2308.818	-2304.1121	-2306.2159	-2308.3609	1
RD	$\hat{\theta}=1.4900$	22.4788	46.9576	47.1798	46.2586	45.1862	4

Table 5 also shows the parameter estimates to each one of the four fitted distributions for the second data set (data set II), the table also provide the values of $-ll$, AIC , BIC , $CAIC$ and $HQIC$ of the fitted models. The values in **Table 5** indicate that the TRD has better performance with the lowest values of AIC , $CAIC$,

BIC and $HQIC$ followed by the $GWRD$, WRD and RD . The secret behind this performance is that, the second data set has a higher degree of skewness and kurtosis meanwhile, our proposed model has various shapes with a moderate peak and skewness.

Table 6: Performance of the distribution using the AIC , $CAIC$, BIC and $HQIC$ values of the models based on data set III.

Distributions	Parameter estimates	$-ll$ ($-\log$ -likelihood value)	AIC	$CAIC$	BIC	$HQIC$	Ranks of models performance
$GWRD$	$\hat{\theta}=0.1090$ $\hat{\alpha}=0.8462$ $\hat{\beta}=5.0865$	50.0391	106.0782	106.4653	105.5368	101.6379	1
WRD	$\hat{\theta}=0.1197$ $\hat{\alpha}=1.3299$ $\hat{\beta}=1.3024$	86.5747	179.1494	179.5365	178.6080	174.7092	2
TRD	$\hat{\theta}=1.3024$ $\hat{\alpha}=0.3480$ $\hat{\lambda}=-0.9587$	88.8744	181.7488	181.9393	181.3879	178.7887	3
RD	$\hat{\theta}=1.2382$	98.2084	198.4168	198.4793	198.2363	196.9367	4

Similarly, **Table 6** also presents the parameter estimates and the values of $-LL$, AIC , BIC , $CAIC$ and $HQIC$ for the four fitted models. The values in the above table also provide evidence that the $GWRD$ has better performance with the lowest values of AIC , $CAIC$, BIC and $HQIC$ compared to the other three models. This also implies that the $GWRD$ could be used to model all kinds of data sets since the above data set is approximately normal. It also shows that the $GWRD$ has various shapes as earlier stated and proven by its graph of pdf and other properties.

CONCLUSION

In this article, a new distribution has been proposed. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done appropriately. Some plots of the distribution revealed that it can take any shape depending on values of the parameters. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the Generalized Weibull-Rayleigh distribution could be used to model age-dependent events or variables whose survival decreases as time grows or where survival rate decreases with time. The results of the three applications showed that the proposed distribution (Generalized Weibull-Rayleigh distribution) performs better than the Weibull-Rayleigh, transmuted Rayleigh and the Rayleigh distributions irrespective of the nature of the data sets. This implies that the proposed distribution can be used when we have skewed and symmetric datasets.

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