

ON DISTINCT FUZZY SUBGROUPS OF NON-TRIVIAL SEMI-DIRECT PRODUCT OF $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

By

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E-mail: olayiwola.a@jsu.edu.ng**ABSTRACT**

In this paper, we compute the number of distinct fuzzy subgroups of Non-trivial direct semi-product of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ with respect to an equivalence relation existing in literature. Our computation shows that $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ has 112 distinct fuzzy subgroups.

Keywords: Fuzzy subgroups; Equivalence relation;

INTRODUCTION

Counting fuzzy subgroups of finite groups is a fundamental problem of fuzzy group theory. Research had focused on counting distinct fuzzy subgroups of finite groups with respect to an equivalence relation \sim [1,2,3], however the number of distinct fuzzy subgroups of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ has not being counted. This paper was therefore designed to compute the number of distinct fuzzy subgroups of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$.

In classical group theory, counting the number of subgroups of a finite group is not difficult as there are existing literatures that can assist in such regard. Without any equivalence relations, the number of fuzzy subgroups of any finite group is infinite even for the

trivial group $\{e\}$, so in order to count the number of distinct fuzzy subgroups it makes sense to define an equivalence relation on the fuzzy subgroup. For other equivalence relations see [4][5][6][7].

Preliminaries

Let G be a group and $F(G)$ be the collection of all fuzzy subset of G . An element μ of $F(G)$ is said to be fuzzy subgroup of G if satisfies the following conditions.

- (i) $\mu(xy) \geq \min(\mu(x), \mu(y))$ for all $x, y \in G$
- (ii) $\mu(x^{-1}) \geq \mu(x)$ for any $x \in G$.

Theorem [1]

The function $\sim : G \rightarrow [0,1]$ is called a fuzzy subgroup of G if there is a chain $P_1 \prec P_2 \prec \dots \prec P_n = G$ in

subgroup lattice of G such that \sim be written as $\sim(x) = \begin{cases} \Gamma_1, & x \in P_1 \\ \Gamma_2, & x \in P_2 - P_1 \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_n, & x \in P_n - P_{n-1} \end{cases}$

In what follows we give the equivalence relation used in this paper. For details see [1]

Definition [1]

Let \sim and x be fuzzy subgroups of G of the form

$$\sim(x) = \begin{cases} \Gamma_1, & x \in P_1 \\ \Gamma_2, & x \in P_2 - P_1 \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_n, & x \in P_n - P_{n-1} \end{cases}, x(x) = \begin{cases} \Gamma_1, & x \in M_1 \\ \Gamma_2, & x \in M_2 - M_1 \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_m, & x \in M_m - M_{m-1} \end{cases}$$

Then we say that \sim and x are equivalent and write $\sim \sim x$, if

- (i) $m = n$ and
- (ii) $P_i = M_i, \forall i \in \{1,2,\dots,m\}$. This relation is indeed an equivalence relation.

Example 3.2. [1] Consider the group $G = \mathbb{Z}_{12}$. Let \sim, x, r, s be functions from \mathbb{Z}_{12} in to $[0,1]$ such that

$$\sim(x) = \begin{cases} 1, & x \in \{0,2,4,6,8,10\} \\ \frac{1}{2}, & x \in \{1,3,5,7,9,11\} \end{cases}, x(x) = \begin{cases} 1, & x \in \{0,1,2,3,4,5\} \\ \frac{1}{2}, & x \in \{6,7,8,9,10,11\} \end{cases}$$

$$r(x) = \begin{cases} 1, & x \in \{0, 4, 8, \} \\ \frac{1}{2}, & x \in \{2, 6, 10\} \\ \frac{1}{3}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases}, s(x) = \begin{cases} \frac{1}{2}, & x \in \{0, 4, 8, \} \\ \frac{1}{4}, & x \in \{2, 6, 10\} \\ 0, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases}$$

It is observed that $P_1(\sim) = \{0, 2, 4, 6, 8, 10\}, P_2(\sim) = \{\mathbb{Z}_{12}\}$. Meaning $P_1(\sim), P_2(\sim)$ are both subgroups of \mathbb{Z}_{12} . By Theorem 2.2, \sim is a fuzzy subgroup of $P_1(\sim) = \{0, 2, 4, 6, 8, 10\}, P_2(\sim) = \{\mathbb{Z}_{12}\}$. In a

similar manner r and s are fuzzy subgroups of \mathbb{Z}_{12} . However, $|r| \neq |s|$ and $|\sim| \neq |s|$, by definition 2.3 we have \sim is equivalent with r and s . But $|r| = |s|$ and it is clear that $P_i(r) = P_i(s), \forall i \in \{1, 2, 3\}$. Hence $r \sim s$

Lemma [1]

The number of fuzzy subgroups of G is equal to the number of chains on the subgroups lattice of G .

Proof: see [1]

The number of fuzzy subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$.

The group $\mathbb{Z}_4 \times \mathbb{Z}_4$ is well known in literature, it has presentation

$$\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle x, y \mid x^4 = y^4 = e, yxy^{-1} = x^3 \rangle$$

where x, y are its generators. It has fifteen (15)

$$D = \langle a \rangle, E = \langle b \rangle, F = \langle a^2, b^2 \rangle, H = \langle a^2b, b^2 \rangle, J = \langle ab, b^2 \rangle, K = \langle a^3b, b^2 \rangle,$$

$$L = \langle a^2, ab^2 \rangle$$

It also has three (3) subgroups of order two (2) namely $M = \langle a^2 \rangle, N = \langle b^2 \rangle, O = \langle a^2b^2 \rangle$.

The other two subgroups are trivial subgroups, that is $\mathbb{Z}_4 \times \mathbb{Z}_4$ itself and $\{e\}$. We shall count the fuzzy subgroups from the lattice subgroups diagram below $\mathbb{Z}_4 \times \mathbb{Z}_4$ and also by applying Lemma [3.4]. Clearly there are five (5) subgroups on the maximal chain of $\mathbb{Z}_4 \times \mathbb{Z}_4$. As a result, the fuzzy subgroup \sim of $\mathbb{Z}_4 \times \mathbb{Z}_4$ length 1, 2, 3, 4 or 5. Let \sim be a fuzzy

subgroups. Three (3) subgroups are of order 8, namely; $A = \langle a^2, b \rangle, B = \langle a^2, ab, b^2 \rangle, C = \langle a, b^2 \rangle$. It has seven (7) subgroups of order four (4) namely;

subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$, then every \sim can be identified as $P_1(\sim)$. If $P_1(\sim) = \mathbb{Z}_4 \times \mathbb{Z}_4$, we have only one (1) fuzzy subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$, that is $\sim_1(x) = 1, \forall x \in \mathbb{Z}_4 \times \mathbb{Z}_4$. If $P_1(\sim) = A$, the only option we have is $P_2(\sim) = \mathbb{Z}_4 \times \mathbb{Z}_4$, that is just one fuzzy subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$ for $P_1(\sim) = A$. That is

$$\sim_2(x) = \begin{cases} \mu_1, & x \in A \\ \mu_2 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - A \end{cases}$$

In a similar manner, we have one fuzzy subgroup for $P_1(\sim) = B$ and $P_1(\sim) = C$. If $P_1(\sim) = D$, then we have two chains, which are $D \prec \mathbb{Z}_4 \times \mathbb{Z}_4$ and $D \prec C \prec \mathbb{Z}_4 \times \mathbb{Z}_4$

Therefore, we get two fuzzy subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ with $P_1(\sim) = D$, which are

$$\sim_3(x) = \begin{cases} \mu_1, & x \in D \\ \mu_2 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - D \end{cases} \quad \sim_4(x) = \begin{cases} \mu_1, & x \in A \\ \mu_2 & x \in C \setminus D \\ \mu_3 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - C \end{cases}$$

Similarly, we have two (2) fuzzy subgroups for

$P_1(\sim) = E, P_1(\sim) = H, P_1(\sim) = J, P_1(\sim) = K, P_1(\sim) = L$. If $P_1(\sim) = F$, then we have four (4) chains, which are

$F \prec \mathbb{Z}_4 \times \mathbb{Z}_4, F \prec A \prec \mathbb{Z}_4 \times \mathbb{Z}_4, F \prec B \prec \mathbb{Z}_4 \times \mathbb{Z}_4, F \prec C \prec \mathbb{Z}_4 \times \mathbb{Z}_4$. Therefore, we have four (4) distinct fuzzy subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ with $P_1(\sim) = F$. They are given by

$$\sim_{10}(x) = \begin{cases} \mu_1, & x \in F \\ \mu_2 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - F \end{cases} \quad \sim_{11}(x) = \begin{cases} \mu_1, & x \in F \\ \mu_2 & x \in A \setminus F \\ \mu_3 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - A \end{cases}$$

$$\sim_{12}(x) = \begin{cases} \mu_1, & x \in F \\ \mu_2 & x \in B \setminus F \\ \mu_3 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - B \end{cases} \quad \sim_{13}(x) = \begin{cases} \mu_1, & x \in F \\ \mu_2 & x \in C \setminus F \\ \mu_3 & x \in (\mathbb{Z}_4 \times \mathbb{Z}_4) - C \end{cases}$$

Following a similar procedure, we have

- (1) Two (2) distinct fuzzy subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ for $P_1(\sim) = H, P_1(\sim) = J, P_1(\sim) = K, P_1(\sim) = L$.
- (2) Twelve (12) fuzzy distinct subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ for $P_1(\sim) = M$; Sixteen (16) fuzzy

subgroups for $P_1(\sim) = N$; and two fuzzy

Eight (8) subgroups for $P_1(\sim) = O$;

- (3) Fifty-six (56) distinct fuzzy subgroups for $P_1(\sim) = \{e\}$. Hence the total number of distinct fuzzy subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4 = 1+1+1+2+2+2+2+2+4+12+16+8+56=112$.

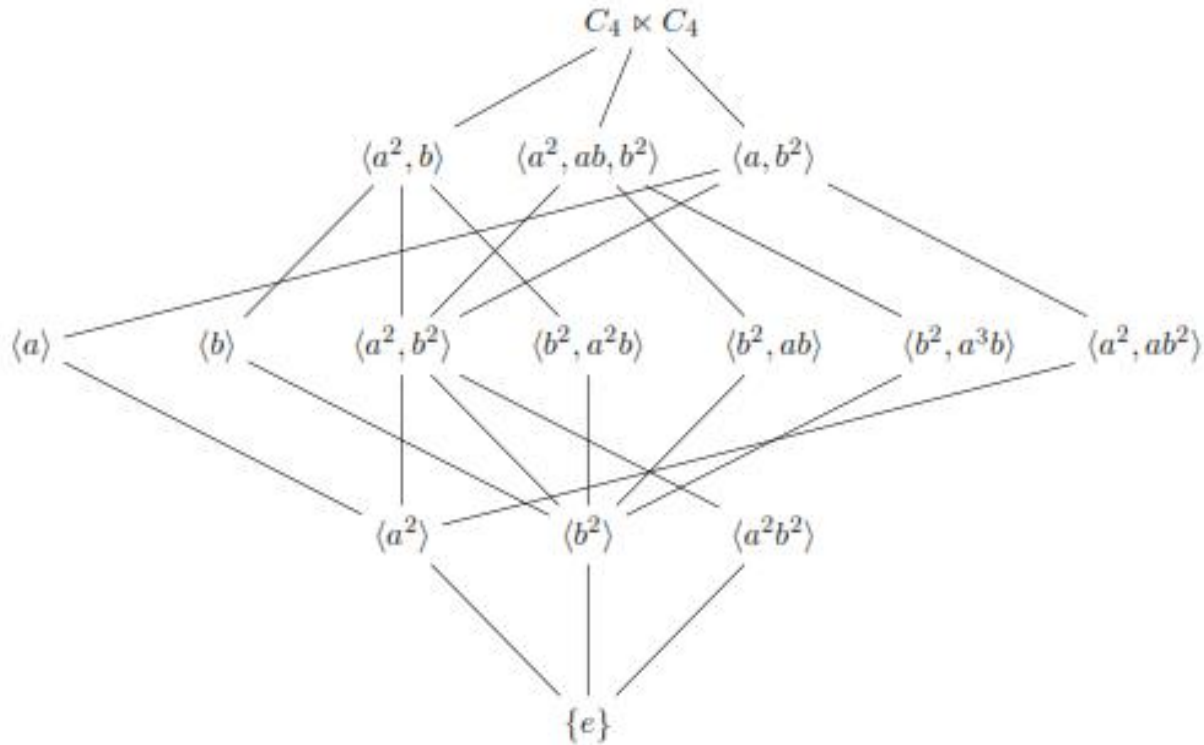


Figure 1: Subgroup Lattice of $Z_4 \times Z_4$

CONCLUSION

Our result has shown that the subgroup lattice in very important in counting distinct fuzzy subgroups. The procedure is cumbersome and time consuming. However, a more efficient tool may be designed for this counting. This will definite constitute a subject of future research.

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