

ON SUBGROUPS LATTICE OF SOME \mathcal{A}_2 -GROUPS

By

Abdulhakeem Olayiwola and Abiola Deborah Akinremi

Department of Mathematics & Computer Science

Sule Lamido Univeresity, Kafin Hausa, Jigawa State

Department of Mathematics, University of Ibadan

Email: olayiwola.a@jsu.edu.ng, aamiisreal77@gmail.com

Abstract

A p -group G is said to be an \mathcal{A}_n -group if it contains a non-Abelian subgroup of index p^{n-1} but all its subgroups of index p^n are Abelian. Every non-Abelian group of order p^4 is either an \mathcal{A}_1 -group or \mathcal{A}_2 -group. For $p = 2$, there are 14 groups of order 16, 5 are Abelian, 3 are \mathcal{A}_1 -group and 6 are \mathcal{A}_2 -group. The purpose of this study is to give all subgroups of \mathcal{A}_2 -group of order 16 and present subgroup lattice diagrams.

Key Words: p -group, \mathcal{A}_n -group, \mathcal{A}_2 -group, Subgroup Lattice

INTRODUCTION

The theory of lattice and subgroups lattice of a group G is very important and has many interesting applications. This lively branch of mathematics can be traced back to the work of Oystein Ore (1937,1938), where he proved that a group G is locally cyclic if and if its lattice subgroups is distributive. If in addition the lattice satisfies the ascending chain condition, then the group is cyclic. Another interesting application can be found in fuzzy groups. The number of chain of subgroup lattice of group G and the number of fuzzy subgroups of a group are closely related, see [2], [3], [4], [5], [6] and [7]. In this work, we present the subgroup lattice diagram of all \mathcal{A}_2 -groups for $p = 2$. Such diagrams can assist in counting the fuzzy subgroups of these \mathcal{A}_2 -groups.

Preliminaries

In this section, we give presentations of the \mathcal{A}_2 -groups for $p = 2$, some basic definitions and theorems that will be used in our main results. A p -group G is said to be an \mathcal{A}_n -group if it contains a nonabelian subgroup of index p^{n-1} but all its subgroups of index p^n

are abelian [8]. Furthermore, every nonabelian group of order p^4 is either an \mathcal{A}_1 - or \mathcal{A}_2 -group [8]. Therefore, in what follows, we let $p = 2, n = 4$ so $|G|=2^4$. The presentations of all \mathcal{A}_2 -group of order 2^4 is given below;

1. $D_8 = \langle a, b | a^8 = b^2 = 1, a^b = a^{-1} \rangle$
2. $SD_8 = \langle a, b | a^8 = b^2 = 1, a^b = a^3 \rangle$
3. $Q_8 = \langle a, b | a^8 = 1, a^4 = b^2, a^b = a^{-1} \rangle$
4. $D_8 \times C_2 = \langle a, b, c | a^4 = b^2 = c^2 = 1, a^b = a^{-1}, ac = ca, bc = cb \rangle$
5. $D_8 * C_4 = \langle a, b, c | a^4 = b^2 = 1, a^2 = c^2, a^b = a^{-1}, bc = cb, ac = ca \rangle$
6. $D_8 \times C_2 = \langle a, b, c | a^4 = c^2 = 1, a^2 = b^2, a^b = a^{-1}, ac = ca, bc = cb \rangle$

Definition

A partially ordered set or POSET is a set P and binary relation \leq such that for all $a, b, c \in P$

1. $a \leq a$ (Reflexive)
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (Transitive)
3. $a \leq b$ and $b \leq a$ implies $a = b$ (Symmetric)

A pair of elements a, b is comparable if $a \leq b$ or $b \leq a$ otherwise they are incomparable. A POSET without incomparable elements is called a totally ordered. A non-empty subset S of P is a chain in P if S is totally ordered by \leq .

Definition

Let P, \leq be a poset and let $S \subseteq P$. An upper bound for S is an element $x \in P$ for which $s \leq x$, for all $s \in S$. The least upper bound of S is called the supremum or join of S . A lower bound for S is an element $x \in P$ for which $x \leq s$, for all $s \in S$. The greatest lower bound of S is called the infimum or meet of S . POSET P, \leq is called a lattice if x, y elements of P has a supremum and an infimum. The set of all subgroups of G under the "subgroup" relation is a lattice. This lattice is called the lattice subgroup of G .

Theorem (Lagrange Theorem) [1]

If H is a subgroup of a finite group G , then the order H divides the order of G and $|G| = |H| |G/H|$.

Theorem [7]

If G is a finite group and $a \in G$, then the order of a is a divisor of order of G .

MAIN RESULTS

The group $D_6 = \langle a, b | a^3 = b^2 = 1, a^2 = a^{-1} \rangle$ is an \mathcal{A}_2 -group, the elements and order of each element is as presented below:

Order	Elements
1	$\{e\}$
2	$a^3, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b$
4	a^2, a^5
8	a, a^5, a^7

Obviously, the trivial subgroups are D_6 itself and the identity $\{e\}$. By Lagrange's theorem, all non-trivial subgroups of D_6 can have order 2, 4, or 8. We determine all these subgroups and use it to construct the diagram of lattice subgroup of the group.

Subgroups of Order 2 for D_6

Let M be a subgroup of order 2. Since 2 is a prime number, then M is cyclic. Thus, M is generated by an element of D_6 of order 2. Hence, all subgroups of order 2 are $\langle a^4 \rangle = \{e, a^4\}$.

$\langle b \rangle = \{e, b\}, \langle ab \rangle = \{e, ab\}, \langle a^2b \rangle = \{e, a^2b\}, \langle a^3b \rangle = \{e, a^3b\}, \langle a^4b \rangle = \{e, a^4b\}, \langle a^5b \rangle = \{e, a^5b\}, \langle a^6b \rangle = \{e, a^6b\}, \langle a^7b \rangle = \{e, a^7b\}$. As a result, we obtain 9 subgroups of order 2 for D_6 .

Subgroups of Order 4 for D_6

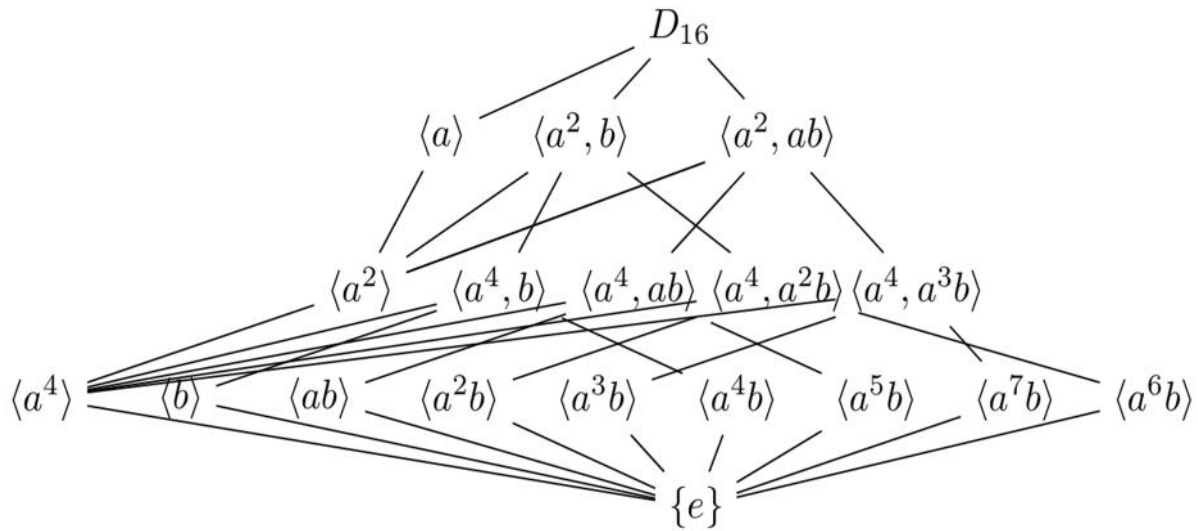
Let K be arbitrary subgroup of D_6 of order 4. By theorem 1.5, the elements of K must have order 1, 2 or 4. If K consist of an element of order 4, then K is cyclic its generator is $\langle a^2 \rangle = \{e, a^2, a^4, a^6\}$. If K is generated by elements of order 2, then K does not have elements of

order 4. By taking the products of all elements of order 2, we found all subgroups of order 4 generated by elements of order 2 are

$\langle a^4, b \rangle = \{e, a^4, b, a^4b\}, \langle a^4, ab \rangle = \{e, a^4, ab, a^5b\}, \langle a^4, a^2b \rangle = \{e, a^4, a^2b, a^6b\}, \langle a^4, a^3b \rangle = \{e, a^4, a^3b, a^7b\}$. Hence, we have 5 subgroups of order 4 D_6 .

Subgroups of Order 8 for D_6

Let P be an arbitrary subgroup of order 8, if P contains an element of order 8, then P is cyclic. The generator of this subgroup is $\langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$. To get other subgroups of order 8, we take all possible products of elements of order 2 and order 4. From these products we get $\langle a^2, b \rangle = \{e, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}, \langle a^2, ab \rangle = \{e, a^2, a^4, a^6, ab, a^3b, a^5b, a^7b\}$. Thus, we have 3 subgroups of order 8. From this result, we have the diagram of lattice subgroups of D_6 which is given below. In what follows, we use a similar argument to obtain subgroups of other groups and we present them as follows:



Subgroups of Order 2 for SD_6

Using a similar argument we obtain subgroups of order 2 for SD_6 , which are $\langle a^4 \rangle = \{e, a^4\}$, $\langle b \rangle = \{e, b\}$, $\langle e, a^2b \rangle = \{e, a^2b\}$, $\langle e, a^4b \rangle = \{e, a^4b\}$, $\langle e, a^6b \rangle = \{e, a^6b\}$. From this, we get 5 subgroups of order 2 for SD_6 .

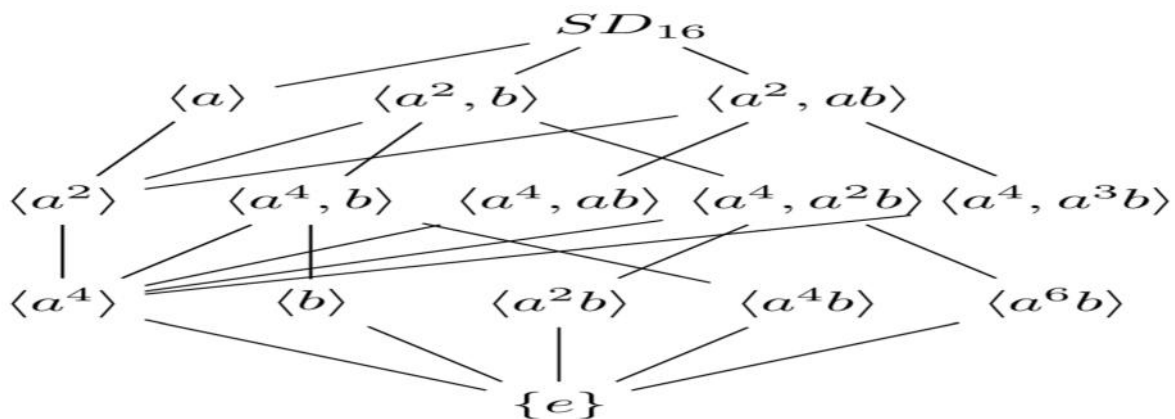
Subgroups of Order 4 for SD_6

The subgroups of order 4 for SD_6 are also obtained in a similar way and are given by: $\langle a^2 \rangle = \{e, a^2, a^4, a^6\}$.

$\langle a^4, b \rangle = \{e, a^4, b, a^4b\}$, $\langle a^4, a^2b \rangle = \{e, a^4, a^2b, a^6b\}$, $\langle a^4, ab \rangle = \{e, a^4, ab, a^5b\}$, $\langle a^4, a^3b \rangle = \{e, a^4, a^3b, a^7b\}$. Hence, we have 5 subgroups of order 4.

Subgroups of Order 8 for SD_6

The subgroups of order 8 for SD_6 are 3 and are given by, $\langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$, $\langle a^2, b \rangle = \{e, a^2, a^4, a^6, a^2b, a^4b, a^6b\}$, $\langle a^2, ab \rangle = \{e, a^2, a^4, a^6, ab, a^3b, a^5b, a^7b\}$.



Subgroup of Order 2 for D_6

The subgroup of order 2 is given by $\langle a^3 \rangle = \{e, a^3\}$

Subgroups of Order 4 for D_6

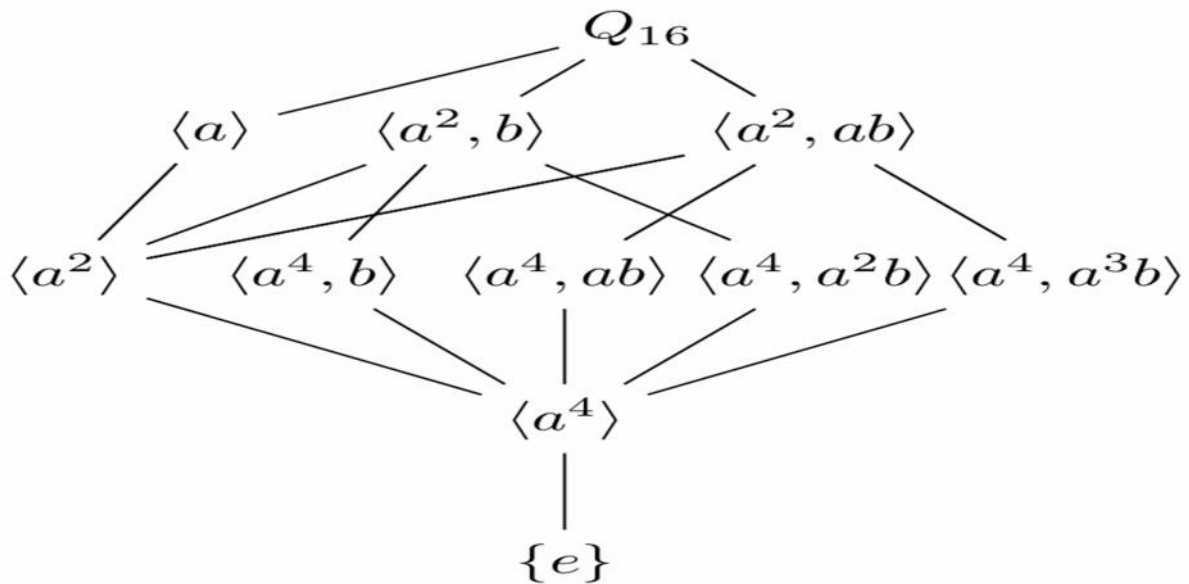
There are 5 subgroups of order 4 for D_6 and they are

- given by $\langle a^2 \rangle = \{e, a^2, a^4, a^6\}$, $\langle a^3, b \rangle = \{e, a^3, b, a^5 b\}$,
- $\langle a^3, ab \rangle = \{e, a^3, ab, a^5 b\}$, $\langle a^3, a^2 b \rangle = \{e, a^3, a^2 b, a^5 b\}$,
- $\langle a^3, a^3 b \rangle = \{e, a^3, a^3 b, a^5 b\}$

3.9 Subgroups of Order 8 for D_6

We obtain 3 subgroups of order 8 for D_6 and they are

- given by: $\langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$,
- $\langle a^2, b \rangle = \{e, a^2, a^4, a^6, b, a^2 b, a^4 b, a^6 b\}$,
- $\langle a^2, ab \rangle = \{e, a^2, a^4, a^6, ab, a^3 b, a^5 b, a^7 b\}$



Subgroups of Order 2 for $D_8 \times C_2$

There are 11 subgroups of order 2 of $D_8 \times C_2$ and they are given by:

- $\langle a^2 \rangle = \{e, a^2\}$, $\langle b \rangle = \{e, b\}$, $\langle ab \rangle = \{e, ab\}$, $\langle a^2 b \rangle = \{e, a^2 b\}$, $\langle a^3 b \rangle = \{e, a^3 b\}$,
- $\langle c \rangle = \{e, c\}$, $\langle a^2 c \rangle = \{e, a^2 c\}$, $\langle bc \rangle = \{e, bc\}$, $\langle abc \rangle = \{e, abc\}$,
- $\langle a^2 bc \rangle = \{e, a^2 bc\}$, $\langle a^3 bc \rangle = \{e, a^3 bc\}$

Subgroups of Order 4 for $D_8 \times C_2$

We obtain 15 subgroups of order 4 of $D_8 \times C_2$, these are

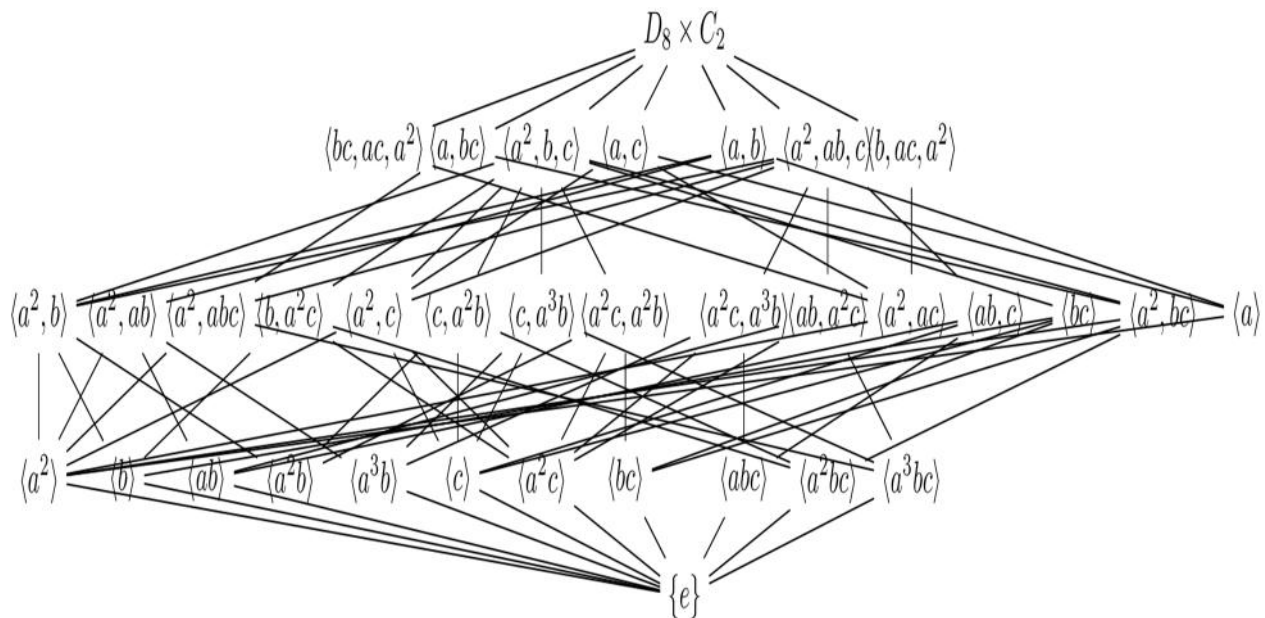
- given by: $\langle e, a \rangle = \{e, a, a^2, a^3\}$, $\langle a^2, b \rangle = \{e, a^2, b, a^2 b\}$,
- $\langle a^2, ab \rangle = \{e, a^2, ab, a^3 b\}$, $\langle a^2, c \rangle = \{e, a^2, c, a^2 c\}$,
- $\langle a^2, bc \rangle = \{e, a^2, bc, a^3 bc\}$, $\langle a^2, abc \rangle = \{e, a^2, abc, a^3 bc\}$,
- $\langle b, c \rangle = \{e, b, c, bc\}$, $\langle b, a^2 c \rangle = \{e, b, a^2 c, a^2 bc\}$,
- $\langle ab, c \rangle = \{e, ab, c, abc\}$, $\langle a^2 b, c \rangle = \{e, c, a^2 b, a^2 bc\}$,

- $\langle a^3 b, c \rangle = \{e, c, a^3 b, a^3 bc\}$, $\langle a^2 c, a^2 b \rangle = \{e, a^2 c, a^2 b, bc\}$,
- $\langle a^2 c, a^3 b \rangle = \{e, a^2 c, a^3 b, abc\}$, $\langle ab, a^2 c \rangle = \{e, ab, a^2 c, a^3 bc\}$,
- $\langle a^2, ac \rangle = \{e, a^2, ac, a^3 c\}$.

Subgroups of Order 8 for $D_8 \times C_2$

There are 7 subgroups of order 8 for $D_8 \times C_2$ and they are given by:

- $\langle a, b \rangle = \{e, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$,
- $\langle a, c \rangle = \{e, a, a^2, a^3, c, ac, a^2 c, a^3 c\}$,
- $\langle a^2, b, c \rangle = \{e, a^2, b, c, a^2 b, a^2 c, bc, a^2 bc\}$,
- $\langle a^2, ab, c \rangle = \{e, a^2, ab, c, a^3 b, a^2 c, abc, a^3 bc\}$,
- $\langle a, bc \rangle = \{e, a, a^2, a^3, bc, abc, a^2 bc, a^3 bc\}$,
- $\langle b, ac, a^2 \rangle = \{e, b, ac, a^2, abc, a^2 b, a^3 c, a^3 bc\}$,
- $\langle bc, ac, a^2 \rangle = \{e, ac, bc, a^2, a^3 c, a^2 bc, ab, a^3 b\}$,



Subgroups of Order 2 for $D_8 * C_4$

There are 7 subgroups of order 2 for $D_8 * C_4$ the central product of D_8 and C_4 and they are $\langle a^2 \rangle = \{e, a^2\}$, $\langle b \rangle = \{e, b\}$, $\langle ab \rangle = \{e, ab\}$, $\langle a^2b \rangle = \{e, a^2b\}$, $\langle a^3b \rangle = \{e, a^3b\}$, $\langle ac \rangle = \{e, ac\}$, $\langle a^3c \rangle = \{e, a^3c\}$.

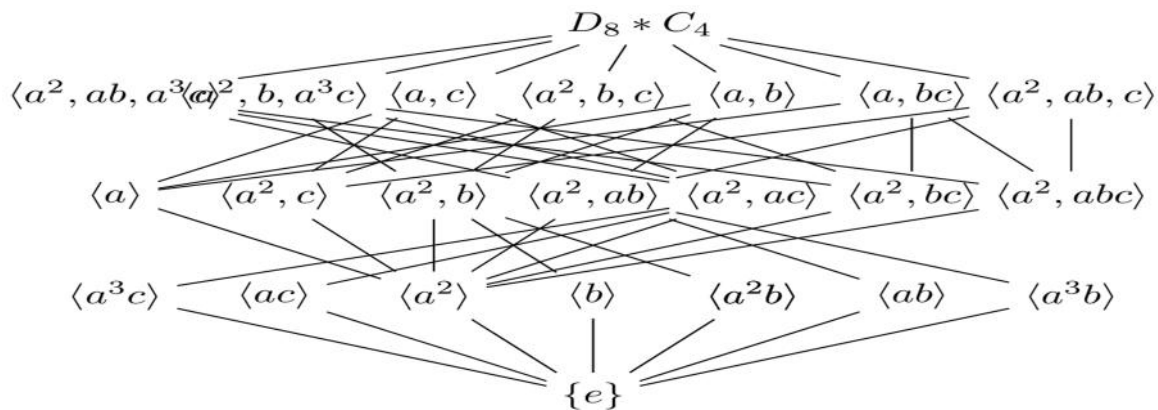
Subgroups of Order 4 for $D_8 * C_4$

There are 7 subgroups of order 4 of $D_8 * C_4$ which are given by: $\langle a \rangle = \{e, a, a^2, a^3\}$, $\langle a^2, b \rangle = \{e, a^2, b, a^2b\}$, $\langle a^2, ab \rangle = \{e, a^2, ab, a^3b\}$, $\langle a^2, ac \rangle = \{e, a^2, ac, a^3c\}$, $\langle a^2, c \rangle = \{e, a^2, c, a^2c\}$, $\langle a^2, bc \rangle = \{e, a^2, bc, a^2bc\}$,

$\langle a^2, abc \rangle = \{e, a^2, abc, a^3bc\}$

3.15 Subgroups of Order 8 for $D_8 * C_4$

We obtain 7 subgroups of order 8 for $D_8 * C_4$ and are given by: $\langle a, b \rangle = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, $\langle a, c \rangle = \{e, a, a^2, a^3, c, ac, a^2c, a^3c\}$, $\langle a, bc \rangle = \{e, a, abc, a^2bc, a^3bc, a^2, a^3\}$, $\langle a^2, b, c \rangle = \{e, a^2, b, c, a^2b, a^2c, a^2bc, bc\}$, $\langle a^2, b, a^3c \rangle = \{e, a^2, b, a^2b, a^3c, ac, a^3bc, abc\}$, $\langle a^2, ab, c \rangle = \{e, a^2, ab, a^3b, abc, c, a^2c, a^3bc\}$, $\langle a^2, ab, a^3c \rangle = \{e, a^2, ab, a^3b, a^3c, ac, bc, a^2bc\}$



Subgroups of Order 2 for $Q_8 \times C_2$

We obtain 3 subgroups of order 2 for the group $Q_8 \times C_2$ given by: $\langle a^2 \rangle = \{e, a^2\}, \langle c \rangle = \{e, c\}, \langle a^2 c \rangle = \{e, a^2 c\}$

Subgroups of Order 4 for $Q_8 \times C_2$

There are 7 subgroups of order of 4 for the group $Q_8 \times C_2$ and these are given by: $\langle a \rangle = \{e, a, a^2, a^3\},$

$\langle a^2, c \rangle = \{e, a^2, c, a^2 c\}, \langle a^2, b \rangle = \{e, a^2, b, a^2 b\},$

$\langle a^2, ab \rangle = \{e, a^2, ab, a^3 b\}, \langle a^2, ac \rangle = \{e, a^2, ac, a^3 c\},$

$\langle a^2, bc \rangle = \{e, a^2, bc, a^3 bc\}, \langle a \rangle = \{e, a, a^2, a^3\}$

Subgroups of Order 8 for $Q_8 \times C_2$

We obtain 7 subgroups of order 8 for the group $Q_8 \times C_2$ given by: $\langle a, b \rangle = \{e, a, a^2, a^3, b, ab, a^2 b, a^3 b\},$

$\langle a, c \rangle = \{e, a, a^2, a^3, c, ac, a^2 c, a^3 c\},$

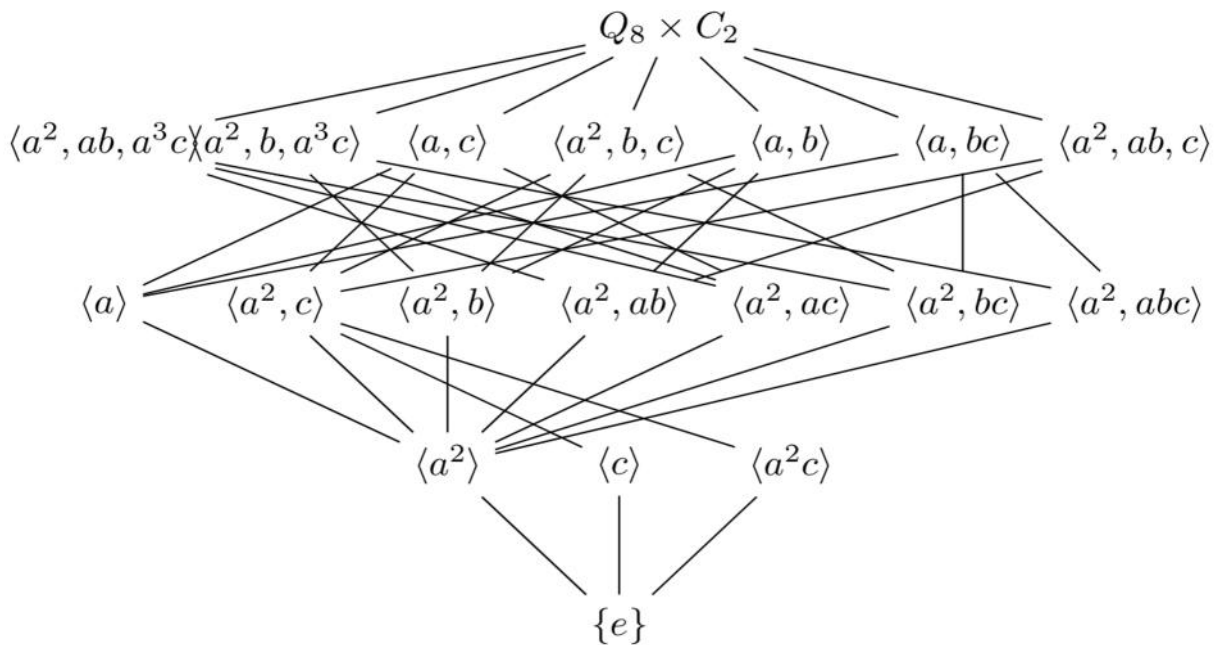
$\langle a^2, b, c \rangle = \{e, a^2, b, a^2 b, c, a^2 c, bc, a^2 bc\},$

$\langle a, a^2 bc \rangle = \{e, a, a^2, bc, a^2 bc, a^3 bc, a^3, abc\},$

$\langle a^2, b, a^3 c \rangle = \{e, a^2, a^2 b, b, a^3 c, ac, abc, a^3 bc\},$

$\langle a^2, ab, c \rangle = \{e, a^2, ab, c, abc, a^3 b, a^2 c, a^3 bc\},$

$\langle a^2, ab, a^3 c \rangle = \{e, a^2, ab, a^3 c, ac, bc, a^2 bc, a^3 b\},$



CONCLUSION

Lattice subgroups are very important in classification of fuzzy subgroups. As the order of the group gets increasingly larger, constructing these diagrams becomes extremely difficult. Although there are existing programs for drawing this diagram, however they are limited to very few classes of finite groups. Developing a programme for such problem will surely be research in future.

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