
Numerical Studies for Solving Linear Fractional Integro-differential Equations Based on
Constructed Orthogonal Polynomials

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ABSTRACT

In this paper, two numerical methods for solving fractional integro-differential equations are presented. The fractional derivative is considered in the Caputo sense. The proposed methods are standard least squares method (SLSM) and modified homotopy perturbation methods (MHPM) with the aids of constructed orthogonal polynomials as basic functions. The suggested methods reduce this type of problem to the solution of system of linear algebraic equations and then solved using MAPLE 18. To demonstrate the accuracy and applicability of the presented methods some numerical examples are provided. Numerical results show that the methods are easy to implement and accurate when applied to fractional integro-differential equations. The graphical solutions of each the method is displayed.

Keywords: Orthogonal polynomials; Numerical studies; Standard least squares; Modified homotopy perturbation.

INTRODUCTION

The fractional calculus has a long history from 1695, when the derivative of order $\alpha = \frac{1}{2}$ has been described by Leibniz [Podlubny (1999), Momani and Qaralleh (2006), Adam (2004) and Fadi et.al (2011)]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. There are many interesting books about fractional calculus and fractional differential equations [Caputo (1967), Samko et.al (1993), Munkhammar (2005), Podlubny (1999)]. The use of fractional differentiation for the mathematical modeling of real world physical problems has been wide spread in recent years, e.g the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc. Derivatives of non-integer order can be defined in different ways, e.g. Riemann Liouville, Grunwald Letnikov, Caputo and Generalized Functions Approach Podlubny (1999). Furthermore, the use of numerical methods for the numerical solution of fractional integro-differential equations cannot be over-emphasized, for it is of great significance to Mathematician, Enginners and Physicists. In recent years, much concentration have been given for the solutions of fractional differentials and integro-differentials equations. Rawashdeh (2006) suggested a numerical solution of fractional integro-differential equations employing collocation method in which polynomial spline functions was used to find the approximate solution.

Momani and Qaralleh (2006) suggested an efficient method for the solution of systems of fractional integro-differential equations using Adomian decomposition method (ADM).

Also, Mittal and Nigam (2008) employed Adomian Decomposition Method for the solution of fractional Integro-differential equations. ADM requires the construction of Adomian polynomials which was reported demanding to construct. Homotopy Analysis methods were applied for the solution of initial value problems of fractional order by Hashim *etal* (2009). These authors decomposed the given problems into basically two parts using linear and nonlinear operators. The basic presumption was that the solutions of the problem could be conveyed as series of polynomials. The truncated parts of these polynomials are then solved to obtain the approximate solutions of the problems. Mohammed (2014) applied least squares method and shifted Chebyshev polynomial for the solution of fractional integro-differential equations. In the work, the author employed shifted Cheyshev polynomial of the first kind $T_n^*(x)$, $n = 0, 1, \dots, n$ as basis function and the result was presented graphically. Oyedepo et al. (2016) proposed numerical methods for the solution fractional integro differential equations using least squares method and Bernstein polynomials as basic functions. In their work, an approximate solution taken together with the Least - Squares method are utilized to reduce the fractional integro-differential equations to system of algebraic equations, which are solved for the unknown constants associated with the approximate solution. Here this study, therefor proposed two numerical methods for solving fractional integro differential equations and to recursively generate orthogonal polynomials which satisfy the interval of consideration that provide less rigorous works in terms of computational cost with improved accuracy using Standard Least Squares Method (SLSM and Modified Homotopy Perturbation Method (MHPM).

The general form of the class of problem considered in this work is given as:

$$D^\alpha u(x) = p(x)u(x) + f(x) + \int_0^1 k(x,t)u(x)dt, \quad 0 \leq x, t \leq 1, \tag{1}$$

With the following supplementary conditions:

$$u^{(i)}(0) = \delta_i, \quad i = 0, 1, 2, \dots, n - 1, \quad n - 1 < \alpha \leq n, \quad n \in N \tag{2}$$

Where $D^\alpha u(x)$ indicates the α th Caputo fractional derivative of $u(x)$; $p(x), f(x), K(x, t)$ are given smooth functions, δ_i are real constant, x and t are real variables varying $[0, 1]$ and $u(x)$ is the unknown function to be determined.

SOME RELEVANT BASIC DEFINITIONS

Definition 1

Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). Example $D^{\frac{1}{2}}, D^\pi, D^{2+i}$ e.t.c

Definition 2

Gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} dt \tag{3}$$

This integral converges when real part of z is positive ($Re(z) > 0$).

$$\Gamma(1 + z) = z\Gamma(z) \tag{4}$$

When z is a positive integer

$$\Gamma(z) = (z - 1)! \tag{5}$$

Definition 3

Beta function is defined as

$$B(v, m) = \int_0^1 (1 - u)^{v-1} u^{m-1} du = \frac{\Gamma(v)\Gamma(m)}{\Gamma(v+m)} = B(v, m), \text{ where } v, m \in R_+ \tag{6}$$

Definition 4

Riemann - Liouville fractional integral is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, x > 0, \tag{7}$$

J^α denotes the fractional integral of order α

Definition 5

Riemann - Liouville fractional derivative denoted D^α is defined as

$$D^\alpha J^\alpha f(x) = f(x) \tag{8}$$

Definition 6

Riemann-Liouville fractional derivative defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s) ds, \tag{9}$$

m is positive integer with the property that $m - 1 < \alpha < m$.

Definition 7

The Caputor Fractional Derivative is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s) ds \tag{10}$$

Where m is a positive integer with the property that $m - 1 < \alpha < m$

For example if $0 < \alpha < 1$ the caputo fractional derivative is

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^1(s) ds \tag{11}$$

Hence, we have the following properties:

- (1) $J^\alpha J^\nu f = J^{\alpha+\nu} f, \alpha, \nu > 0, f \in C_\mu, \mu > 0$
- (2) $J^\alpha x^\gamma = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$
- (3) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k}, \quad x > 0, m - 1 < \alpha \leq m$
- (4) $D^\alpha J^\alpha f(x) = f(x), \quad x > 0, m - 1 < \alpha \leq m,$
- (5) $D^\alpha C = 0, C$ is the constant,
- (6) $\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \dots\}$

Definition 8

Orthogonality: Two functions say $u_p(x)$ and $u_q(x)$ defined on the interval $a \leq x \leq b$ are said to be orthogonal if

$$\langle u_p(x), u_q(x) \rangle = \int_a^b u_p(x) u_q(x) dx = 0 \tag{12}$$

If, on the other hand, a third function $w(x) > 0$ exists such that

$$\langle u_p(x), u_q(x) \rangle = \int_a^b w(x) u_p(x) u_q(x) dx = 0 \tag{13}$$

Then, we say that $u_p(x)$ and $u_q(x)$ are mutually orthogonal with respect to the weight function $w(x)$.

Generally, we write:

$$\int_a^b w(x)u_p(x)u_q(x) = \begin{cases} 0 & p \neq q \\ \int_a^b w(x)u_p^2(x)dx & p = q, \end{cases} \tag{14}$$

Definition 9.

In this work, we defined absolute error as:

$$\text{Absolute Error} = |Y(x) - y_n(x)|; \quad 0 \leq x \leq 1, \tag{15}$$

where $Y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

CONSTRUCTION OF ORTHOGONAL POLYNOMIALS

In this section, we constructed our orthogonal polynomials using the general weight function of the form: $w(x)=(a + bx^i)^k$.

This corresponds to quartic functions for $a = 1, b = -1, k = 1$ and $i = 2$ respectively, satisfying the orthogonality conditions in the interval $[a, b]$ under consideration.

According to Gram-schmidt orthogonalization process, the orthogonal polynomial $u_j(x)$ Valid in the interval $[a, b]$ with the leading term x^j , is given as

$$u_j(x) = x^j - \sum_{i=0}^{j-1} a_{j,i}u_i(x) \quad i = 0,1,2 \dots \dots j - 1 \text{ and } j \geq 1 \tag{16}$$

Where $u_j(x)$ is an increasing polynomial of degree j and $u_i(x)$ are the corresponding values of the approximating functions in x . Then, starting with $u_0(x) = 1$, we find that the linear polynomial $u_j(x)$ with leading term x , is written as

$$u_1(x) = x + a_{1,0}u_0(x) \tag{17}$$

Where $a_{1,0}$ is a constant to be determined. Since $u_1(x)$ and $u_0(x)$ are orthogonal, we have

$$\int_a^b w(x)u_1(x)u_0(x) dx = 0 = \int_a^b xw(x)u_0(x) dx + a_{1,0} \int_a^b w(x)u_0^2(x) dx \tag{18}$$

Using (14) and (18). From the above, we have

$$a_{1,0} = \frac{\int_a^b w(x)xu_0(x)}{\int_a^b w(x)u_0^2(x)dx} \tag{19}$$

Hence, substituting (19) into (17) gives

$$u_1(x) = x + \frac{\int_a^b w(x)xu_0(x)}{\int_a^b w(x)u_0^2(x)dx} \tag{20}$$

Proceeding in this way, the method is generalized and is written as

$$u_j(x) = x^j + a_{j,0}u_0(x) + a_{j,1}u_1(x) + a_{j,2}u_2(x) + \dots a_{j,j-1}u_{j-1}(x) \tag{21}$$

Where the constants $a_{j,0}$ are so chosen such that $u_j(x)$ is orthogonal to $u_0(x), u_1(x), u_2(x) \dots \dots u_{j-1}(x)$. These conditions yield

$$a_{j,i} = - \frac{\int_a^b x^j w(x) u_0(x) dx}{\int_a^b w(x) u_0^2(x) dx} \tag{22}$$

For $k = 1, a = 1, b = -1$ and $i = 4$ valid in $[0, 1]$

$$w(x) = 1 - x^2 \tag{23}$$

$$u_0(x) = 1 \tag{24}$$

We have $k = 1, j = 1$ and $u_0(x) = 1$, we write equation (16) as

$$u_1(x) = x - a_{1,0} u_0(x) \tag{25}$$

Simplifying the above equation we have

$$u_1(x) = x, \quad u_2(x) = x^2 - \frac{1}{5} \tag{26}$$

The shifted equivalent of the (26) that is valid in $[0, 1]$ are given as:

$$u_0^*(x) = 1, u_1^*(x) = 2x - 1, u_2^*(x) = 4x^2 - 4x + \frac{4}{5} \tag{27}$$

In this work the method assumed an approximate solution with the orthogonal polynomial as basis function as

$$u(x) \cong u_n(x) = \sum_{i=0}^n a_i u_i^*(x) \tag{28}$$

Where $u_i^*(x)$ denotes the orthogonal polynomial of degree N where $a_i, i = 0, 1, 2, \dots$ are constants.

DEMONSTRATION OF THE PROPOSED METHODS.

In this section, we demonstrated the two proposed methods mentioned above

Standard Least Squares Method (SLSM).

The standard least squares method with the constructed orthogonal polynomials as basic function is applied to find the numerical solution of fractional Integro-differential equation of the type in (1) and (2). This method is based on approximating the unknown function $u(x)$ by assuming an approximation solution of the form defined in (28)

Consider equation (1) operating with J^α on both sides as follows:

$$J^\alpha D^\alpha u(x) = J^\alpha f(x) + J^\alpha \left(\int_0^1 k(x,t) u(t) dt \right) \tag{29}$$

$$u(x) = \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} + J^\alpha f(x) + J^\alpha \left[\int_0^1 k(x,t) u(t) dt \right] \tag{30}$$

Substituting (28) into (30)

$$\sum_{i=0}^n a_i u_i^*(x) = \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} + J^\alpha f(x) + J^\alpha \left[\int_0^1 k(x,t) \sum_{i=0}^n a_i u_i^*(t) dt \right] \tag{31}$$

Hence, the residual equation is obtained as

$$R(a_0, a_1, \dots, a_n) = \sum_{i=0}^n a_i u_i^*(x) - \left\{ \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} + J^\alpha f(x) + J^\alpha \left[\int_0^1 k(x, t) \sum_{i=0}^n a_i u_i^*(t) dt \right] \right\} \tag{32}$$

Let

$$S(a_0, a_1, \dots, a_n) = \int_0^1 [R(a_0, a_1, \dots, a_n)]^2 w(x) dx \tag{33}$$

Where $w(x)$ is the positive weight function defined in the interval, $[a, b]$. In this work, we take $w(x) = 1$ for simplicity. Thus,

$$S(a_0, a_1, \dots, a_n) = \int_0^1 \left\{ \sum_{i=0}^n a_i u_i^*(x) - \left\{ \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} + J^\alpha f(x) + \left[\int_0^1 k(x, t) \sum_{i=0}^n a_i u_i^*(t) dt \right] \right\} \right\}^2 dx \tag{34}$$

In order to minimize equation (34), we obtained the values of $a_i (i \geq 0)$ by finding the minimum value of S as:

$$\frac{\partial S}{\partial a_i} = 0, i = 0, 1, 2, \dots, n \tag{35}$$

Applying (35) on (34), we have

$$\int_0^1 \left\{ \sum_{i=0}^n a_i u_i^*(x) - \left\{ \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} + J^\alpha f(x) + J^\alpha \left[\int_0^1 k(x, t) \sum_{i=0}^n a_i u_i^*(t) dt \right] \right\} \right\} dx \times \int_0^1 \left\{ u_i^*(x) - J^\alpha \left(\int_0^1 k(x, t) u_i^*(t) dt \right) \right\} dx \tag{36}$$

Thus, (36) are then simplified for $i = 0, 1, \dots, n$ to obtain $(n + 1)$ algebraic system of equations in $(n + 1)$ unknown a'_i s which are put in matrix form as follow:

$$A = \begin{pmatrix} \int_0^1 R(x, a_0) h_0 dx & \int_0^1 R(x, a_1) h_0 dx & \dots & \int_0^1 R(x, a_n) h_0 dx \\ \int_0^1 R(x, a_0) h_1 dx & \int_0^1 R(x, a_1) h_1 dx & \dots & \int_0^1 R(x, a_n) h_1 dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 R(x, a_0) h_n dx & \int_0^1 R(x, a_1) h_n dx & \dots & \int_0^1 R(x, a_n) h_n dx \end{pmatrix},$$

B=

$$\begin{pmatrix} \int_0^1 \left[J^\alpha f(x) + \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} \right] h_0 dx \\ \int_0^1 \left[J^\alpha f(x) + \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} \right] h_1 dx \\ \vdots \\ \int_0^1 \left[J^\alpha f(x) + \sum_{k=0}^{m-1} u^k(0^+) \frac{x^k}{k!} \right] h_n dx \end{pmatrix} \tag{37}$$

Where

$$h_i = u_i^*(x) - J^\alpha \left[\int_0^1 k(x, t) u_i^*(t) dt \right], i = 0, 1, \dots, n$$

$$R(x, a_i) = \sum_{i=0}^n a_i u_i^*(x) - J^\alpha \left[\int_0^1 k(x, t) \sum_{i=0}^n a_i u_i^*(t) dt \right], i = 0, 1, \dots, n$$

The $(n + 1)$ linear equation are then solved using Gaussian elimination method or any suitable computer package like maple i8 to obtain the unknown constants $a_i (i = 0(1)n)$, which are then substituted back into the assumed approximate solution to give the required approximation solution.

MODIFIED HOMOTOPY PERTURBATION METHOD (MHPM).

Consider (i) operating with J^α on both sides as follows:

$$J^\alpha [D^\alpha u(x)] = J^\alpha [p(x)u(x) + f(x) + \int_0^x k(x,t)u(t)dt] \tag{38}$$

$$u(x) = \sum_k^{m-1} u^k(o^+) \frac{x^k}{k!} + J^\alpha [p(x)u(x) + f(x) + \int_0^x k(x,t)u(t)dt] \tag{39}$$

We solved (39) by the homotopy perturbation method, we constructed the following homotopy:

$$H(U(x),p) = (1-p)(U(x) - u_0(x)) + p[U(x) - \sum_k^{m-1} u^k(o^+) \frac{x^k}{k!} - J^\alpha [p(x)u(x) - f(x) - \int_0^x k(x,t)u(t)dt] \tag{40}$$

Where $p \in [0,1]$ is called the homotopy parameter, and $u_0(x)$ is an initial approximation to the solution of (40). Simplifying (40), to get

$$U(x) = u_0(x) - pu_0(x) + p \left[\sum_k^{m-1} u^k(o^+) \frac{x^k}{k!} \right] + J^\alpha [p(x)U(x)] + p \left[J^\alpha f(x) + J^\alpha \left(\int_0^x k(x,t)u(t)dt \right) \right] \tag{41}$$

Suppose the solution of (41) have the form:

$$U(x) = \sum_{i=0}^n p^i U_i(x) \tag{42}$$

Where $U_i(x)$, $i = 0,1,2,3, \dots$ are functions to be determined?

Now suppose that the initial approximation to the solution $u_0(x)$ has the form:

$$u_0(x) = \sum_{i=0}^n a_i u_i^*(x) \tag{43}$$

Where a_i are unknown coefficient of $u_0(x)$, $u_1(x), u_2(x), \dots$ are specific functions depending on the problem.

Substituting (43) into (41), we have

$$\sum_{i=0}^n p^i U_i(x) = u_0(x) - pu_0(x) + p \left[\sum_k^{m-1} u^k(o^+) \frac{x^k}{k!} \right] + J^\alpha [p(x) \sum_{i=0}^n p^i U_i(x)] + p \left[J^\alpha f(x) + J^\alpha \left(\int_0^x k(x,t) \sum_{i=0}^n p^i U_i(t) dt \right) \right] \tag{44}$$

Equating the like power of p we obtain the following

$$p^0: U_0(x) = u_0(x) \tag{45}$$

$$p^1: U_1(x) = -u_0(x) + \sum_k^{m-1} u^k(o^+) \frac{x^k}{k!} + J^\alpha [p(x)U_0(x)] + J^\alpha f(x) + J^\alpha \left(\int_0^x k(x,t)U_0(t)dt \right) \tag{46}$$

$$p^2: U_2(x) = J^\alpha [p(x)U_1(x)] + J^\alpha \left(\int_0^x k(x,t)U_1(t)dt \right) \tag{47}$$

⋮
⋮
⋮

$$p^{n+1}: U_{n+1}(x) = J^\alpha [p(x)U_n(x)] + J^\alpha \left(\int_0^x k(x,t)U_n(t)dt \right) \tag{48}$$

Now if these equations are solved in a way that $U_1(x) = 0$ then (47 - 48) resulted to $U_2(x) = U_3(x) = \dots = 0$. Therefore the solution can be obtained by substituting (43) into (45) to get

$$U_0(x) = \sum_{i=0}^n a_i u_i^*(x) \tag{49}$$

Also substituting (49) into (46), to have

$$-\sum_{i=0}^n a_i u_i^*(x) + \sum_{k=0}^{m-1} u^k(o^+) \frac{x^k}{k!} + J^\alpha [p(x) \sum_{i=0}^n a_i u_i^*(x)] + J^\alpha f(x) + J^\alpha (\int_0^x k(x,t) \sum_{i=0}^n a_i u_i^*(t) dt) = 0 \tag{50}$$

Simplifying (50) further gives algebraic linear systems of equations by equating corresponding coefficients of various powers of the independent variable in (50). The systems of equations are then solved using Gaussian elimination method or any suitable computer package like maple 18. The results of the unknown constants obtained are then substituted back into (43) to give the required approximate solution.

NUMERICAL EXAMPLES

Here we demonstrated the proposed methods discussed above on some examples. The problems are solved using the constructed orthogonal polynomials with respect to the weight functions in form of quartic for $n = 3$

In this section, we demonstrated the proposed methods discussed above on some examples. All result are obtained using Maple 18

Example 1 by Quadratic Weighted Polynomial (QWP) as basis function for $n = 3$ using Standard Least Squares Method (SLSM).

Example 1.

Consider the following fractional Integro-differential equation:

$$D^{1/2}u(x) = \frac{{}^{(8/3)}x^{3/2}-2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt, \quad 0 \leq x \leq 1, \tag{51}$$

Subject to $u(0) = 0$, with exact solution $U(x) = x^2 - x$ (52)

Applying the above method on (5.1), the required approximate solution is given as:

$$u_3(x) = -0.00000001113121838 - 0.999793803x + 0.9998138319x^2 - 0.00000389462927x^3 \tag{53}$$

Example 1 by Quadratic Weighted Polynomial (QWP) as basis function for $n = 3$ using modified homotopy perturbation method (MHPM).

Example 1.

Consider the following fractional Integro-differential equation:

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Subject to $u(0) = 0$, with exact solution $U(x) = x^2 - x$ (55)

The above method on (51), the required approximate solution is given as:

$$u_3(x) = -0.9997988114x + 0.9997988108x^2 \tag{56}$$

RESULTS.

TABLE 1: Numerical results of Example 1.

X	Exact Solution	Approximate Solution of SLSM. n = 3	Approximate Solution of SLSM. n = 5	Absolute error of SLSM. n = 3	Absolute error of SLSM. n = 5
0.0	0.00	-0.00000011131218	-0.00000003203909	1.113E-07	3.204E-08
0.1	-0.09	-0.08998135725000	-0.08998134362000	1.864E-05	1.866E-05
0.2	-0.16	-0.15996634990000	-0.15996627520000	3.365E-05	3.372E-05
0.3	-0.21	-0.20995511270000	-0.20995492530000	4.489E-05	4.507E-05
0.4	-0.24	-0.23994766900000	-0.23994735930000	5.233E-05	5.264E-05
0.5	-0.25	-0.24994404190000	-0.24994361650000	5.233E-05	5.638E-05
0.6	-0.24	-0.23994425520000	-0.23994371850000	5.574E-05	5.628E-05
0.7	-0.25	-0.20994833220000	-0.20994767650000	5.167E-05	5.232E-05
0.8	-0.16	-0.15995629600000	-0.15995498800000	4.370E-05	4.450E-05
0.9	-0.09	-0.08996816998000	-0.08996719867000	3.183E-05	3.280E-05
1.0	0.00	0.00001602227057	0.00001719808348	1.602E-05	1.720E-05

TABLE 2: Numerical results of Example 1.

X	Exact Solution	Approximate Solution of MHPM. n = 3	Approximate Solution of MHPM. n = 5	Absolute error of MHPM. n = 3	Absolute error of MHPM. n = 5
0.0	0.00	0.00000000000000	0.00000000000000	0.000E+00	0.000E+00
0.1	-0.09	-0.08998189299000	-0.08998189299000	1.811E-5	1.811E-5
0.2	-0.16	-0.15996780980000	-0.15996780980000	3.219E-5	3.219E-5
0.3	-0.21	-0.20995775030000	-0.20995775030000	4.225E-5	4.225E-5
0.4	-0.24	-0.23995171470000	-0.23995171470000	4.829E-5	4.829E-5
0.5	-0.25	-0.24994970280000	-0.24994970380000	5.030E-5	5.030E-5
0.6	-0.24	-0.23995171470000	-0.23995171470000	4.829E-5	4.829E-5
0.7	-0.25	-0.20995775040000	-0.20995775040000	4.225E-5	4.225E-5
0.8	-0.16	-0.15996780990000	-0.15996780990000	3.219E-5	3.219E-5
0.9	-0.09	-0.08998189320000	-0.08998189320000	3.811E-5	3.811E-5
1.0	0.00	-0.08998189380000	-0.0000000020000	2.000E-10	2.000E-10

Example 2.

Consider the following fractional Integro-differential equation:

$$D_{6u}^{\frac{5}{6}}(x) = -\frac{3}{91} \frac{x^{\frac{1}{6}} \Gamma(5/6)(-91 + 216x^2)}{\pi} + (5 - 2e)x + \int_0^1 xe^t u(t) dt, \quad \leq x \leq 1, \quad (57)$$

Subject to $u(0) = 0$ with the exact equation $U(x) = x - x^3$ (58)

TABLE 3: Numerical results of Example 2.

X	Exact Solution	Approximate solution of SLSM. $n = 3$	Approximate solution of SLSM. $n = 5$	Absolute error of SLSM. $n = 3$	Absolute error SLSM. $n = 5$
0.0	0.00	0.00000038280000	0.00000007378081	3.828E-06	7.378E-08
0.1	0.099	0.09895741497000	0.09895750001000	4.259E-05	4.250E-05
0.2	0.192	0.19191319860000	0.19191330470000	8.680E-05	8.670E-05
0.3	0.273	0.27287027400000	0.27287027900000	1.297E-05	1.297E-04
0.4	0.336	0.33583118160000	0.33583110400000	1.688E-04	1.689E-04
0.5	0.375	0.37479846150000	0.37479837500000	2.015E-04	2.016E-04
0.6	0.384	0.38377465420000	0.38377462480000	2.253E-04	2.254E-04
0.7	0.357	0.35676229990000	0.35676234730000	2.377E-04	2.377E-04
0.8	0.288	0.28776393890000	0.28776402140000	2.361E-04	2.360E-04
0.9	0.171	0.17078211170000	0.17078213450000	2.179E-04	2.179E-04
1.0	0.00	0.00018064160000	0.00018079417350	1.806E-04	1.808E-04

TABLE 4: Numerical results of Example 2.

X	Exact Solution	Approximate solution of MHPM. $n = 3$	Approximate solution of MHPM. $n = 5$	Absolute error of MHPM. $n = 3$	Absolute error of MHPM. $n = 5$
0.0	0.00	0.00000000013000	0.00000000013000	1.300E-10	1.300E-10
0.1	0.099	0.09896016869000	0.09896016869000	3.983E-5	3.983E-5
0.2	0.192	0.19192275120000	0.19192275120000	7.725E-5	7.725E-5
0.3	0.273	0.27289016190000	0.27289016190000	1.098E-4	1.098E-4
0.4	0.336	0.33586481460000	0.33586481460000	1.352E-4	1.352E-4
0.5	0.375	0.37484912340000	0.37484912340000	1.509E-4	1.509E-4
0.6	0.384	0.38384550230000	0.38384550230000	1.545E-4	1.545E-4
0.7	0.357	0.35685636540000	0.35685636540000	1.436E-4	1.436E-4
0.8	0.288	0.28788412680000	0.28788412680000	1.159E-4	1.159E-4
0.9	0.171	0.17093120020000	0.17093120020000	6.880E-5	6.880E-5
1.0	0.00	0.00000000010000	0.00000000010000	7.000E-11	7.000E-11

Graphical Representation of Two Methods.

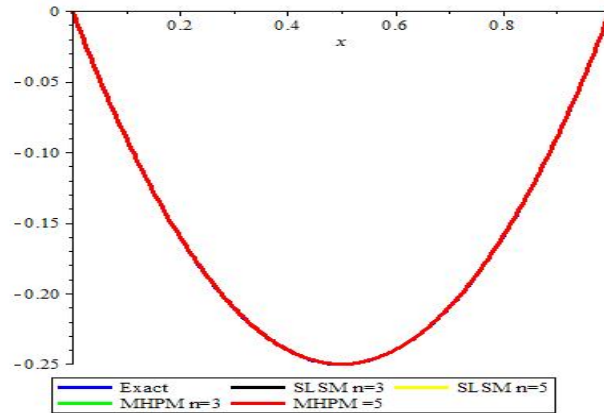


Figure 1 above showing approximate solution of Example 1 for $n = 3$ and 5 using Standard Least Squares Method (SLSM) and Modified Homotopy Perturbation Method (MHPM).

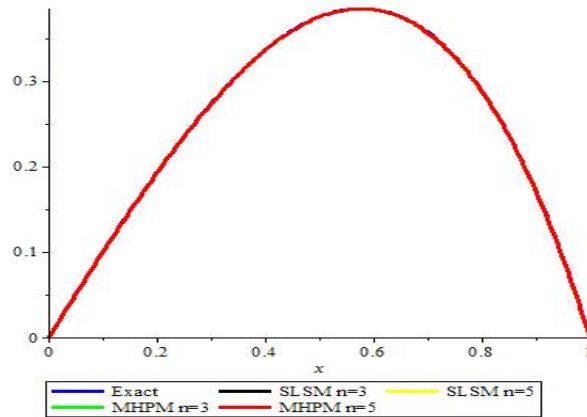


Figure 2 above showing approximate solution of Example 2 for $n = 3$ and 5 using Standard Least Squares Method (SLSM) and Modified Homotopy Perturbation Method (MHPM).

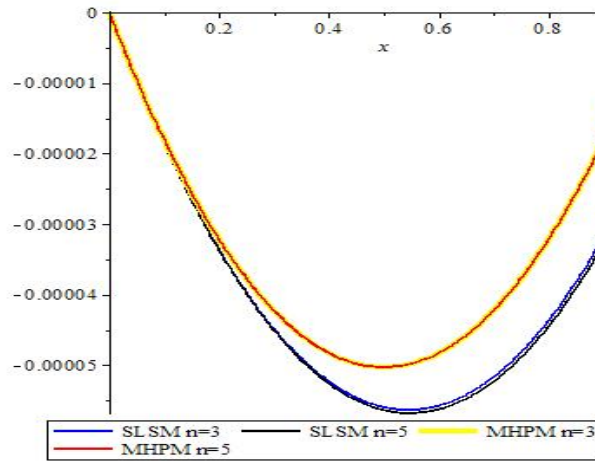


Figure 4 above showing error of Example 1

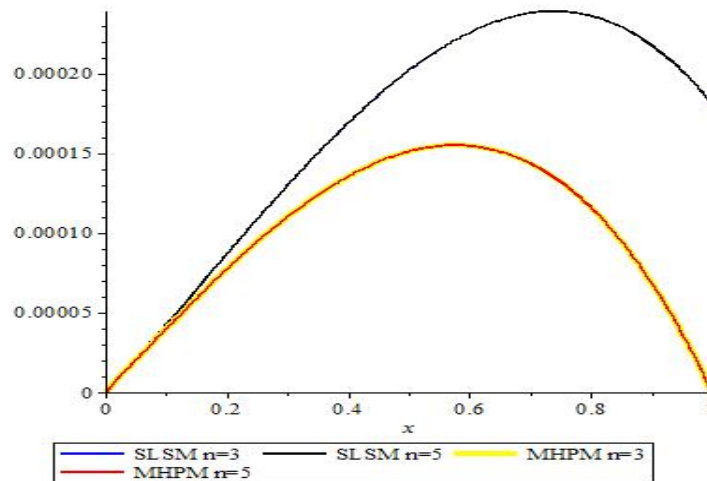


Figure 4 above showing error of Example 2

CONCLUSION

In this work, the study showed that the methods based on constructed orthogonal polynomials were successfully used for solving fractional integro differential equations. We observed that the SLSM involves less computational effort when compared with MHPM. The results obtained showed that the two methods and compared with the exact solutions shown that there is a similarity between the exact and the approximate solution. Calculation showed that SLSM and MHPM are powerful and efficient techniques in finding very good solution for this type of equation. Also the results were presented in graphical forms to further demonstrate the relationship between the methods.

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