

Solution of Fractional Integro-Differential Equation Using Modified Homotopy Perturbation Technique and Constructed Orthogonal Polynomials as Basis Functions

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ABSTRACT

A numerical methodology based on quartic weighted polynomials for finding the solution of fractional integro-differential equations (FIDEs) is presented. The fractional derivative is taken into account within in the Caputo sense. The suggested method involves the application of the homotopy perturbation method and used the initial approximation as the constructed orthogonal polynomials. The ensuing equations involve comparing the coefficients of the homotopy parameter P , which then resulted in a system of a linear algebraic equation and then solved using MAPLE 18. To demonstrate the relevance of the bestowed methodology some numerical examples were solved, and the numerical results obtained show that the techniques are easy to implement and accurate when applied to fractional FIDEs. The graphical solution of the method is displayed.

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INTRODUCTION

According to Leibniz [Adam (2004), Caputo (1967), Momani and Qaralleh (2006) and Samko et. al (1993)], fractional calculus has a long record since 1695, when the derivative of order $\alpha=1/2$ has been reported by Adam (2004). The speculation of derivatives and integrals of non-integer order can be traced back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. There are many fascinating or exciting books about fractional calculus and fractional differential equations (FDEs) [Caputo(1967), Munkhammar (2005) and and Samko et. al (1993)]. Recently fractional calculus has displayed a major role in mathematical modelling of real-world physical problems, e.g. the modelling of the earthquake, reducing the spread of the virus, control the memory behavior of electric socket, etc. Understanding of definitions and benefit of fractional calculus will be made more understandable by quickly discussing some compulsory but relatively simple mathematical definitions that will appear in the study of these concepts. These are The Gamma Function, The Beta Function and Riemann-Liouville Fractional Integral. The help of this class of calculus for solving problems in the branch of Physics, Chemistry, Engineering and Technology has been extensive spread in recent years by many researchers. [Momani and Qaralleh (2006)] Suggested a $D^\alpha u(x) = p(x)u(x) + f(x) + \int_0^x k(x,t)u(x)dt$, $0 \leq x, t \leq 1$,

With the following supplementary conditions:

$$u^{(i)}(0) = \delta_i, \quad i = 0,1,2, \dots, n-1, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}$$

method for the solution of FIDEs called using Adomian decomposition method (ADM). Also Rawashdeh (2006) employed a collocation method for solving fractional FIDEs. Also, ADM was applied by [Mittal, and Nigam(2008)] for the solution of FIDEs. ADM involve the development of Adomian polynomials which are somehow strenuous to get. [Hashim et. al, (2009) and Mohamed et. al (2016)] employed Homotopy Perturbation and Homotopy Analysis methods for solving FIDEs. He (1999) also gives some application of nonlinear FDEs and their approximation. [Taiwo and Odetunde (2011)] applied Iterative Decomposition Method (IDM) for finding a numerical approximation of fractional integro-differential equations. Abdollahpoor (2014) employed the least squares method (LSM) for treating nonlinear fourth order integro-differential equations. [Rajesh et. al (2014)] Applied a method called an efficient method with the help of Bernstein polynomials for solving FDEs. [Mohammed (2014)] employed the least squares method for solving FIDEs using shifted Chebyshev polynomial of the first kind as the basis function. Oyedepo (2016) applied LSM and Bernstein polynomial as the basis function for solving FIDEs. In other to improve on the existing methods we, therefore, produce an alternative method called MHPM based Quartic Weighted Polynomials (QWP) as basis function. The general form of the class of problem considered in this work is given as:

$$(1)$$

$$(2)$$

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Where $D^\alpha u(x)$ indicates the α th Caputo fractional derivative of $u(x)$; $p(x), f(x)$, $K(x, t)$ are given smooth functions, δ_i are real constant, x and t are real variables varying $[0, 1]$ and $u(x)$ is the unknown function to be determined.

SOME RELEVANT BASIC DEFINITIONS

Definition 1.

Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). Example $D^{\frac{1}{2}}, D^\pi, D^{2+i}$ e.t.c

Definition 2.

Gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{3}$$

This integral converges when real part of z is positive ($Re(z) > 0$).

$$\Gamma(1 + z) = z\Gamma(z) \tag{4}$$

When z is a positive integer

$$\Gamma(z) = (z - 1)! \tag{5}$$

Definition 3.

Beta function is defined as

$$B(v, m) = \int_0^1 (1 - u)^{v-1} u^{m-1} du = \frac{\Gamma(v)\Gamma(m)}{\Gamma(v+m)} = B(v, m), \text{ where } v, m \in R_+ \tag{6}$$

Definition 4.

Riemann – Liouville fractional integral is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, x > 0, \tag{7}$$

J^α denotes the fractional integral of order α

Definition 5.

Riemann – Liouville fractional derivative denoted D^α is defined as

$$D^\alpha J^\alpha f(x) = f(x) \tag{8}$$

Definition 6.

Riemann-Liouville fractional derivative defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s) ds, \tag{9}$$

m is positive integer with the property that $m - 1 < \alpha < m$.

Definition 7.

The Caputo Fractional Derivative is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s) ds \tag{10}$$

Where m is a positive integer with the property that $m - 1 < \alpha < m$

For example if $0 < \alpha < 1$ the caputo fractional derivative is

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^1(s) ds \tag{11}$$

Hence, we have the following properties:

- (1) $J^\alpha J^\nu f = J^{\alpha+\nu} f, \alpha, \nu > 0, f \in C_\mu, \mu > 0$
- (2) $J^\alpha x^\gamma = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$
- (3) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0) \frac{x^k}{k!}, \quad x > 0, m - 1 < \alpha \leq m$
- (4) $D^\alpha J^\alpha f(x) = f(x), \quad x > 0, m - 1 < \alpha \leq m,$
- (5) $D^\alpha C = 0, C$ is the constant,
- (6) $\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \dots\}$

Definition 8.

Orthogonality: Two functions say $u_p(x)$ and $u_q(x)$ defined on the interval $a \leq x \leq b$ are said to be orthogonal if

$$\langle u_p(x), u_q(x) \rangle = \int_a^b u_p(x) u_q(x) dx = 0 \tag{12}$$

If, on the other hand, a third function $w(x) > 0$ exists such that

$$\langle u_p(x), u_q(x) \rangle = \int_a^b w(x) u_p(x) u_q(x) dx = 0 \tag{13}$$

Then, we say that $u_p(x)$ and $u_q(x)$ are mutually orthogonal with respect to the weight function $w(x)$.

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Generally, we write:

$$\int_a^b w(x)u_p(x)u_q(x)dx = \begin{cases} 0 & p \neq q \\ \int_a^b w(x)u_p^2(x)dx & p = q, \end{cases} \quad (15)$$

Definition 9.

In this work, we defined absolute error as:
 Absolute Error = $|Y(x) - y_n(x)|$; $0 \leq x \leq 1$,
 where $Y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

Construction of Quartic Weighted Polynomials

In this section, we constructed the orthogonal polynomials using the general weight function of the form: $w(x) = (a + bx^i)^k$.

This corresponds to quartic functions for $a = 1, b = -1, k = 1$ and $i = 4$ respectively, satisfying the orthogonality conditions in the interval $[a, b]$ under consideration.

According to Gram-schmidt orthogonalization process, the orthogonal polynomial

$u_j(x)$ Valid in the interval $[a, b]$ with the leading term x^j , is given as

$$u_j(x) = x^j - \sum_{i=0}^{j-1} a_{j,i}u_i(x) \quad i = 0, 1, 2, \dots, j-1 \text{ and } j \geq 1 \quad (16)$$

Where $u_j(x)$ is an increasing polynomial of degree j and $u_i(x)$ are the corresponding values of the approximating functions in x . Then, starting with $u_0(x) = 1$, we find that the linear polynomial $u_1(x)$ with leading term x , is written as

$$u_1(x) = x + a_{1,0}u_0(x) \quad (17)$$

Where $a_{1,0}$ is a constant to be determined? Since $u_1(x)$ and $u_0(x)$ are orthogonal, we have

$$\int_a^b w(x)u_1(x)u_0(x)dx = 0 = \int_a^b xw(x)u_0(x)dx + a_{1,0} \int_a^b w(x)u_0^2(x)dx \quad (18)$$

Using (14) and (18). From the above, we have

$$a_{1,0} = \frac{\int_a^b w(x)xu_0(x)dx}{\int_a^b w(x)u_0^2(x)dx} \quad (19)$$

Hence, substituting (19) into (17) gives

$$u_1(x) = x + \frac{\int_a^b w(x)xu_0(x)dx}{\int_a^b w(x)u_0^2(x)dx} \quad (20)$$

Proceeding in this way, the method is generalized and is written as

$$u_j(x) = x^j + a_{j,0}u_0(x) + a_{j,1}u_1(x) + a_{j,2}u_2(x) + \dots + a_{j,j-1}u_{j-1}(x) \quad (21)$$

Where the constants $a_{j,i}$ are so chosen such that $u_j(x)$ is orthogonal to $u_0(x), u_1(x), u_2(x) \dots u_{j-1}(x)$. These conditions yield

$$a_{j,i} = -\frac{\int_a^b x^i w(x)u_j(x)dx}{\int_a^b w(x)u_i^2(x)dx} \quad (22)$$

For $k = 1, a = 1, b = -1$ and $i = 4$ valid in $[0, 1]$

$$w(x) = 1 - x^4 \quad (23)$$

$$u_0(x) = 1 \quad (24)$$

We have $k = 1, j = 1$ and $u_0(x) = 1$, we write equation (16) as

$$u_1(x) = x - a_{1,0}u_0(x) \quad (25)$$

Simplifying the above equation we have

$$u_1(x) = x, u_2(x) = x^2 - \frac{5}{21} \quad (26)$$

The shifted equivalent of the (25) that is valid in $[0, 1]$ are given as:

$$u_0^*(x) = 1, u_1^*(x) = 2x - 1, u_2^*(x) = 4t^2 - 4x + \frac{16}{21} \quad (27)$$

In this work the method assumed an approximate solution with the orthogonal polynomial as basis function as

$$u(x) \cong u_n(x) = \sum_{i=0}^n a_i u_i^*(x) \quad (28)$$

Where $u_i^*(x)$ denotes the orthogonal polynomial of degree N where $a_i, i = 0, 1, 2, \dots$ are constants.

DEMONSTRATION OF THE PROPOSED METHOD.

This section demonstrated the new method mentioned above

Modified Homotopy Perturbation Method (MHPM).

Consider (1) operating with J^α on both sides as follows:

$$J^\alpha [D^\alpha u(x)] = J^\alpha [p(x)u(x) + f(x) + \int_0^x k(x,t)u(t)dt] \quad (29)$$

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$$u(x) = \sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + J^\alpha [p(x)u(x) + f(x) + \int_0^x k(x,t)u(t)dt] \quad (30)$$

We solved (4.1) by the homotopy perturbation method, we constructed the following homotopy:

$$H(U(x), p) = (1 - p)(U(x) - u_0(x)) + p[U(x) - \sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} - J^\alpha [p(x)u(x) - f(x) - \int_0^x k(x,t)u(t)dt]] \quad (31)$$

Where $p \in [0,1]$ is called the homotopy parameter, and $u_0(x)$ is an initial approximation to the solution of(31). Simplifying(31), to get

$$U(x) = u_0(x) - pu_0(x) + p \left[\sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + J^\alpha [p(x)U(x)] + p[J^\alpha f(x)] + p[J^\alpha (\int_0^x k(x,t)u(t)dt)] \right] \quad (32)$$

Suppose the solution of (32) have the form:

$$U(x) = \sum_{i=0}^n p^i U_i(x) \quad (33)$$

Where $U_i(x)$, $i = 0,1,2,3, \dots$ are functions to be determined?

Now suppose that the initial approximation to the solution $u_0(x)$ has the form:

$$u_0(x) = \sum_{i=0}^n a_i u_i^*(x) \quad (34)$$

Where a_i are unknown coefficient of $u_0(x)$, $u_1(x), u_2(x), \dots$ are specific functions depending on the problem.

Substituting (33) into (32), to have

$$\sum_{i=0}^n p^i U_i(x) = u_0(x) - pu_0(x) + p \left[\sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + J^\alpha [p(x) \sum_{i=0}^n p^i U_i(x)] + p[J^\alpha f(x)] + p[J^\alpha (\int_0^x k(x,t) \sum_{i=0}^n p^i U_i(t) dt)] \right] \quad (35)$$

Equating the like power of p we obtain the following

$$p^0: U_0(x) = u_0(x) \quad (36)$$

$$p^1: U_1(x) = -u_0(x) + \sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + J^\alpha [p(x)U_0(x)] + J^\alpha f(x) + J^\alpha (\int_0^x k(x,t)U_0(t)dt) \quad (37)$$

$$p^2: U_2(x) = J^\alpha [p(x)U_1(x)] + J^\alpha (\int_0^x k(x,t)U_1(t)dt) \quad (38)$$

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$$p^{n+1}: U_{n+1}(x) = J^\alpha [p(x)U_n(x)] + J^\alpha (\int_0^x k(x,t)U_n(t)dt) \quad (39)$$

Now if these equations are solved in a way that $U_1(x) = 0$ then (37 - 39)

resulted to $U_2(x) = U_3(x) = \dots = 0$. Therefore the solution can be obtained by substituting (34) into (36) to get

$$U_0(x) = \sum_{i=0}^n a_i u_i^*(x) \quad (40)$$

Also substituting (34) and (40) into (37), to have

$$-\sum_{i=0}^n a_i u_i^*(x) + \sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + J^\alpha [p(x) \sum_{i=0}^n a_i u_i^*(x)] + J^\alpha f(x) + J^\alpha (\int_0^x k(x,t) \sum_{i=0}^n a_i u_i^*(t) dt) = 0 \quad (41)$$

Simplifying (41) further gives algebraic linear systems of equations by equating corresponding coefficients of various powers of the independent variable in (41). The systems of equations are then solved using maple 18. The unknown constants obtained are then substituted back into (34) to give the required approximate solution.

Numerical Examples

This section, demonstrated the new method discussed above on the general FIDEs using the quartic weighted polynomials as basis functions for $n = 3$. The examples are solved to illustrate the computational cost accuracy and efficiency of the proposed methods using Maple 18.

Example 1: Consider the following fractional Integro-differential

$$D^{\frac{3}{5}} u(x) = -\frac{x^2 e^x}{5} u(x) + \frac{6x^{2.25}}{\Gamma(3.25)} + e^x \int_0^x tu(t)dt \quad (42)$$

$$\text{Subject to } u(0) = 0. \text{ The exact solution is } U(x) = x^3 \quad (43)$$

The approximate solution of (5.1) is obtained using the new method. The required approximate solution is given as:

$$u(x) = 3 \times 10^{-11} + x^3 \quad (44)$$

TABLE 1. Numerical results of Example 1.

x	Exact Solution	Approximate Solution of New Method	Approximate Solution of Method Rawashdeh (2006)	Absolute Error of New method	Absolute Error of Method Rawashdeh (2006)
0.0	0.000	0.00000000003000	0.0000300	3.000E-11	3.000E-5
0.1	0.001	0.00100000003000	-	3.000E-11	-
0.2	0.008	0.00800000003000	0.0080371	3.000E-11	3.710E-5
0.3	0.273	0.02700000003000	-	3.000E-11	-
0.4	0.064	0.06400000003000	0.064024	3.000E-11	2.400E-5
0.5	0.125	0.12500000000000	-	0.00E+00	-
0.6	0.216	0.21600000000000	0.216084	0.00E+00	8.400E-5
0.7	0.343	0.34300000000000	-	0.00E+00	-
0.8	0.512	0.51200000000000	0.512043	0.00E+00	4.300E-5
0.9	0.729	0.72900000000000	-	0.00E+00	-
1.0	1.000	1.00000000000000	1.000028	0.00E+00	2.800E-5

Example 2: Consider the following fractional Integro-differential

$$D^{\frac{1}{2}}u(x) = u(x) + \frac{8x^{2.25}}{3\Gamma(0.5)} - x^2 - \frac{1}{2}x^3 + \int_0^x tu(t)dt \quad (45)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^2$ (46)

The approximate solution of (5.4) is obtained using the new method. The required approximate solution is given as:

$$u(x) = 1 \times 10^{-10} + x^2 \quad (47)$$

TABLE 2: Numerical results of Example 2.

x	Exact Solution	Approximate Solution of New Method	Approximate solution of Method Mohamed et. al (2016)	Absolute Error of New method	Absolute Error of Method Mohamed et. al (2016)
0.0	0.00	0.00000000010000	0.00000000000000	1.000E-10	0.000E+00
0.1	0.01	0.01000000008000	0.00999098347500	8.000E-11	9.017E-06
0.2	0.04	0.04000000006000	0.03984425053000	6.000E-11	1.557E-04
0.3	0.09	0.09000000004000	0.08915051900000	4.000E-11	8.495E-04
0.4	0.16	0.16000000000000	0.15711315420000	0.000E+00	2.887E-03
0.5	0.25	0.25000000000000	0.24243543660000	0.000E+00	7.565E-03
0.6	0.36	0.36000000000000	0.34319333660000	0.000E+00	1.681E-02
0.7	0.49	0.49000000000000	0.45669292160000	0.000E+00	1.681E-02
0.8	0.64	0.63999999900000	0.57931151210000	6.000E-11	6.069E-02
0.9	0.81	0.80999999900000	0.70632167110000	8.000E-11	1.037E-01
1.0	1.00	0.99999999900000	0.83169710000000	1.000E-10	1.683E-01

Example 3: Consider the following fractional Integro-differential

$$D^{\frac{1}{2}}u(x) = (\cos(x) - \sin(x)) u(x) + f(x) + \int_0^x x \sin(t) u(t)dt \quad (48)$$

$$f(x) = \frac{2x^{1.5}}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)}x^{0.5} + x(\cos(x) - x \sin(x) + x^2 \cos(x)) \quad (49)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^2 + x$

The approximate solution of (48) is obtained using the new method. The required approximate solution is given as:

$$u(x) = -3 \times 10^{-10} + 0.999999994x + x^2 \quad (50)$$

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TABLE 3: Numerical results of Example 3.

x	Exact Solution	Approximate Solution New Method	Approximate Solution of Method [Mohamed et. al (2016)]	Absolute Error of New Method	Absolute Error of Method [Mohamed et. al (2016)]
0.0	0.000	- 0.00000000030000	0.000286	3.000E-10	0.000E+00
0.1	0.110	0.10999999910000	-	9.000E-10	-
0.2	0.240	0.23999999850000	0.240010	1.500E-09	1.000E-5
0.3	0.390	0.38999999790000	-	2.100E-09	-
0.4	0.560	0.55999999730000	0.560087	2.700E-09	8.700E-5
0.5	0.750	0.74999999670000	-	3.300E-09	-
0.6	0.960	0.95999999610000	0.960053	3.900E-09	5.300E-5
0.7	1.190	1.18999999600000	-	4.500E-09	-
0.8	1.440	1.43999999500000	1.440081	5.100E-09	8.100E-05
0.9	1.710	1.70999999400000	-	5.700E-09	-
1.0	2.000	1.99999999400000	2.000617	6.300E-09	6.170E-5

Graphical Representation of the Method

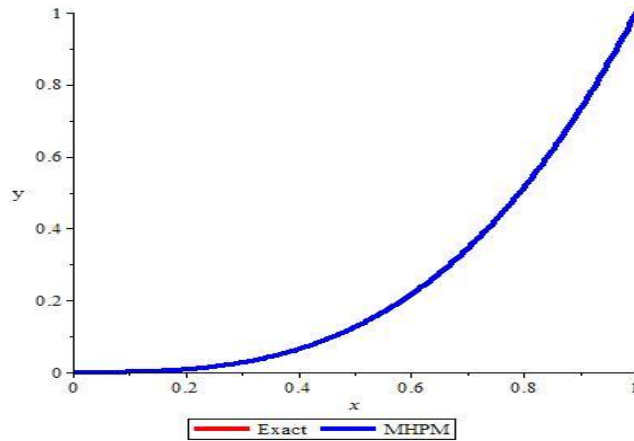


Figure 1: Showing the graph of approximation solution and exact of example 1

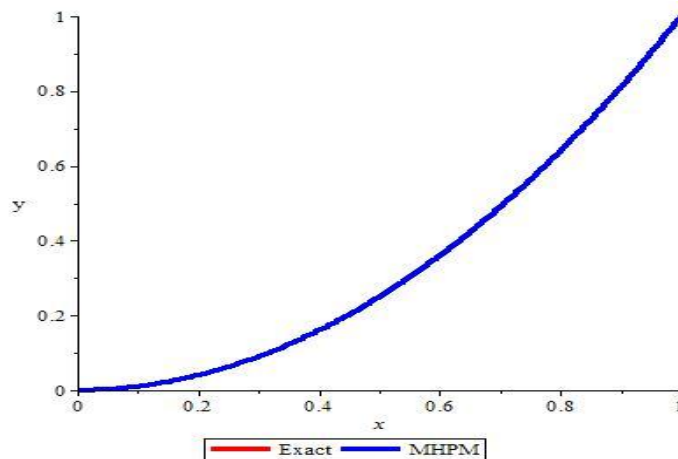


Figure 2: Showing the graph of approximation solution and exact of example 2

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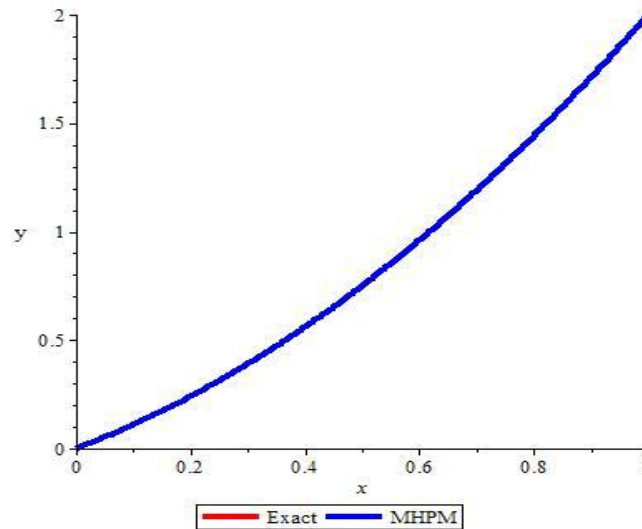


Figure 3: Showing the graph of approximation solution and exact of example 3

DISCUSSION OF RESULT

All the problems presented in this study were solved using maple 18. Table 1 for problem 1 shows that the new method in this study is more accurate than the method of Rawashdeh (2006). Table 2 for problem 2 reveals that the new method with the quartic weighted polynomials performed better than the method of [Mohamed et. al (2016)]. Also Table 3 for problem 3, a comparison was made with the method of Rawashdeh (2006), where again the proposed method was seen to be better in terms of accuracy. It is to be noted that these comparison were made for only those values that are available in the existing literature. The graphs in figures 1 – 5 are presented to further buttress the above observation. However, it was clear that errors of the new method is smaller than that of Rawashdeh (2006) and Mohamed et. al (2016)

CONCLUSION

The study showed that the new method with via quartic weighted polynomials as basis function is successfully used for solving FIDEs. The obtained results, when compared with the existing methods reveal that the new method performed more accurately than the existing methods in the literature. Calculation showed the new method is powerful and effective in finding a good approximate solution for this type of equation. More also, it accelerate the rate of convergence. Also the results were presented in graphical forms to further demonstrate of the method.

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