



Bernstein Collocation Method for the Solution of Fractional Integro-differential Equations

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ABSTRACT

The main purpose of this study is to present an approximation method based on the Bernstein polynomials for fractional integro-differential equations. This method transforms the integro-differential equation to a system of linear algebraic equations by using the collocation points, which is then solved using MAPLE 18. Besides, some examples are presented to illustrate the accuracy of the method and the results were presented in graphical form.

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INTRODUCTION

Fractional calculus is a field dealing with integral and derivatives of arbitrary orders, and their applications in science, engineering and other fields. The idea is from the ordinary calculus. According to Leibniz [Adam (2004), Caputo (1967), Momani and Qaralleh (2006) and Samko et. al (1993)], It was discovered by Leibniz in the year 1695 few years he discovered ordinary calculus by later forgotten due to the complexity of the formula. Many real-world physical problems can be models by fractional integro-differential equations e.g the modeling of the earthquake, reducing the spread of the virus, control the memory behaviour of electric socket and many others. There are many fascinating or exciting books about fractional calculus and fractional differential equations [Caputo(1967), Munkhammar (2005) and Samko et-al(1993)]. Since the fractional calculus has attracted much more interest among mathematicians and other scientists, the solutions of the fractional differential and integro-differential equations have been studied

frequently in recent years. Many FIDEs cannot be solved analytically, and hence finding good approximate solutions, using numerical techniques, will be very helpful. Several numerical methods to solve the FIDEs have been given. The author in [Mittal and Nigam (2008)] applied the Adomian decomposition method (ADM) for the solution of FIDEs. Polynomial spline function was introduced in Rawashdeh (2006) for solving FIDEs. Cubic B- spline wavelets was introduced in Khowsrow et. al (2013) for the numerical solution of FIDEs. Mohamed et-al (2016) employed homotopy analysis transform method for solving FIDEs. Reference Taiwo et. al. (2015) used Perturbed Chebyshev Polynomials for solving FIDEs. In this work, an approximate solution taken together with the Least - Squares method (LSM) is utilized to reduce the fractional integro-differential equations to a system of algebraic equations, which are solved for the unknown constants associated with the approximate solution. Momani et. al (2006) applied an efficient method for finding the solution of systems of



fractional Integro-differential equations. Oyedepo et. al (2016) employed a method called numerical studies for solving fractional FIDEs using Least Squares Method and Bernstein Polynomials. Oyedepo et. al (2019) applied Homotopy perturbation and LSM for solving FIDEs. Construction of orthogonal polynomials was introduced by Oyedepo et. al. (2019) for the solution of FIDEs. Mohammed

(2014) employed LSM for solving FIDEs using shifted Chebyshev polynomial of the first kind as the basis function. In other to improve on the existing methods in the literature, we therefore suggested a method called Bernstine Collocation Method with the aid of Bernstein Polynomials for solving FIDEs. The general form of the class of problem considered in this work is given as:

$$D^\alpha u(x) = p(x)u(x) + f(x) + \int_0^x k(x,t)u(x)dt, \quad 0 \leq x, t \leq 1, \quad (1)$$

With the following supplementary conditions:

$$u^{(j)}(0) = \delta_j, j = 0, 1, 2, \dots, m-1, m-1 < \alpha \leq m, m \in N \quad (2)$$

Where $D^\alpha u(x)$ indicates the α th Caputo fractional derivative of $u(x)$; $p(x), f(x), K(x, t)$ are given smooth functions, δ_j are real constant, x and t are real variables varying $[0, 1]$ and $u(x)$ is the unknown function to be determined.

Some relevant basic definitions

Definition 1

Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). Example $D^{\frac{1}{2}}, D^\pi, D^{2+i}$ e.t.c

Definition 2

Gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (3)$$

This integral converges when real part of z is positive ($Re(z) > 0$).

$$\Gamma(1+z) = z\Gamma(z) \quad (4)$$

When z is a positive integer

$$\Gamma(z) = (z-1)! \quad (5)$$

Definition 3

Beta function is defined as

$$B(v, m) = \int_0^1 (1-u)^{v-1} u^{m-1} du = \frac{\Gamma(v)\Gamma(m)}{\Gamma(v+m)} = B(v, m), \text{Where } v, m \in R_+ \quad (6)$$

Definition 4

Riemann – Liouville fractional integral is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, x > 0, \quad (7)$$

J^α denotes the fractional integral of order α

Definition 5

Riemann – Liouville fractional derivative denoted D^α is defined as

$$D^\alpha J^\alpha f(x) = f(x) \quad (8)$$

Definition 6

Riemann-Liouville fractional derivative defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} f^n(s) ds, \quad (9)$$

m is positive integer with the property that $m-1 < \alpha < m$.

Definition 7

The Caputor Fractional Derivative is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} f^m(s) ds \quad (10)$$

Where m is a positive integer with the property that $n-1 < \alpha < n$

For example if $0 < \alpha < 1$ the caputo fractional derivative is



$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^1(s) ds \quad (11)$$

Hence, we have the following properties:

- (1) $J^\alpha J^\nu f = J^{\alpha+\nu} f, \alpha, \nu > 0, f \in C_\mu, \mu > 0$
- (2) $J^\alpha x^\gamma = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$
- (3) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0) \frac{x^k}{k!}, \quad x > 0, n-1 < \alpha \leq n$
- (4) $D^\alpha J^\alpha f(x) = f(x), \quad x > 0, n-1 < \alpha \leq n,$
- (5) $D^\alpha C = 0, C$ is the constant,
- (6)
$$\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \dots\}$

Definition 8

Bernstein basis polynomials: A Bernstein polynomial [8] of degree N is defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i} \quad i = 0, 1, \dots, m, \quad (12)$$

where,

$$\binom{m}{i} = \frac{m!}{i!(m-i)!} \quad (13)$$

Often, for mathematical convenience, we set $B_{i,m}(x) = 0$ if $i < 0$ or $j > m$

Definition 9

Bernstein polynomials: A linear combination Bernstein basis polynomials

$$u_m(x) = \sum_j^m a_j u_j(x) \quad (14)$$

The Bernstein polynomial of degree n where $a_j, j = 0, 1, 2, \dots, m$ are constants

Examples

The first few Bernstein basis polynomials are:

$$u_0(x) = 1, u_1(x) = a_0(1-x) + a_1x, u_2(x) = a_0(1-2x+x^2) + a_1(2x-2x^2) + a_2x^2$$

Definition 10

In this work, we defined absolute error as:

$$\text{Absolute Error} = |U(x) - u_m(x)|; \quad 0 \leq x \leq 1, \quad (15)$$

where $U(x)$ is the exact solution and $u_m(x)$ is the approximate solution.

Where $u_m(x)$ Bernstein polynomial of degree m where $a_j, j = 0, 1, 2, \dots$ are constants.

DEMONSTRATION OF THE SUGGESTED METHOD

In this section, we demonstrated the proposed method mentioned above

Bernstein Collocation Method (BCM)

The suggested technique via Bernstein polynomials as basis function is applied to find the numerical solution of

fractional integro-differential equation of the type in (1) and (2). This method is based on approximating the unknown function $u(x)$ by assuming an approximation solution of the form defined in (14) which is then substituted in (1) to get:

$$D^\alpha [\sum_j^m a_j u_j(x)] = p(x) [\sum_j^m a_j u_j(x) + f(x) + \int_0^x k(x,t) \sum_j^m a_j u_j(t) dt] \quad (16)$$

Thus, (16) is further expanded to get:

$$a_0 D^\alpha(u_0(x)) + a_1 D^\alpha(u_1(x)) + a_2 D^\alpha(u_2(x)) + \dots + a_m D^\alpha(u_m(x)) =$$



$$p(x)[a_0(u_0^*(x)) + a_1(u_1^*(x)) + a_2(u_2^*(x)) + \dots + a_M(u_M^*(x))] + f(x) + \int_0^x k(x,t)[a_0(u_0^*(t)) + a_1(u_1^*(t)) + a_2(u_2^*(t)) + \dots + a_M(u_M^*(t))]dt \quad (17)$$

Applying the fractional differential operator, D^α on (17) leads to:

$$a_0 \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} + a_1 \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} + \dots + a_M \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} = p(x)[a_0(u_0^*(x)) + a_1(u_1^*(x)) + a_2(u_2^*(x)) + \dots + a_M(u_M^*(x))] + f(x) + \int_0^x k(x,t)[a_0(u_0^*(t)) + a_1(u_1^*(t)) + a_2(u_2^*(t)) + \dots + a_M(u_M^*(t))]dt \quad (18)$$

From equation (18), collecting the like terms leads to

$$a_0 \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} + p(x)(u_0^*(x)) + \int_0^x k(x,t)(u_0^*(t))dt \right] + a_1 \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} + p(x)(u_1^*(x)) + \int_0^x k(x,t)(u_1^*(t))dt \right] + \dots + a_M \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} + p(x)(u_M^*(x)) + \int_0^x k(x,t)(u_M^*(t))dt \right] = -f(x) \quad (19)$$

Where β indicate the power of x the independent terms of the orthogonal polynomials. After evaluating the integral in (19). The resulting equation is then collocated at equally spaced point $x_i = a + \frac{(b-a)i}{M}$, ($i = 1(1)M$) gives linear system algebraic of equations in (M) unknown constants a_i s. Additional

equations are obtained from (3), which are represented in matrix form. The system of equations is then solved using maple 18. The unknown constants obtained are then substituted back into the assumed approximate solution to get the required approximate solution.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & A_{24} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & A_{m4} & \dots & A_{mn} \\ A_{11}^0 & A_{12}^0 & A_{13}^0 & A_{14}^0 & \dots & A_{1n}^0 \\ A_{21}^1 & A_{22}^1 & A_{23}^1 & A_{24}^1 & \dots & A_{2n}^0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ A_{m1}^{n-1} & A_{m2}^{n-1} & A_{m3}^{n-1} & A_{m4}^{n-1} & \dots & A_{mn}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} B_{11} \\ B_{21} \\ \vdots \\ \vdots \\ B_{mn} \\ B_{11}^0 \\ B_{22}^1 \\ \vdots \\ \vdots \\ B_{mn}^{n-1} \end{pmatrix} \quad (20)$$

where A_{is}^1 and A_{is} are the coefficients of a_{is} given as $A_{11} = \frac{\Gamma(\beta+1)}{(\beta-\alpha+1)} x_1^{\beta-\alpha} - p(x)(u_0^*(x_1)) - \int_0^x k(x_1,t)(u_0^*(t))dt$, $A_{12} = \frac{\Gamma(\beta+1)}{(\beta-\alpha+1)} x_2^{\beta-\alpha} - p(x)(u_1^*(x_2)) - \int_0^x k(x_2,t)(u_1^*(t))dt$, $A_{13} = \frac{\Gamma(\beta+1)}{(\beta-\alpha+1)} x_3^{\beta-\alpha} - p(x)(u_2^*(x_3)) - \int_0^x k(x_3,t)(u_2^*(t))dt$, $A_{11}^0 = u_0^0(0)$, $A_{12}^0 = u_0^1(0)$, $\dots, A_{1n}^0 = u_n^0(0)$, $A_{21}^1 = u_0^1(0)$, $A_{22}^1 = u_0^1(0)$, $\dots, A_{2n}^0 = u_n^0(0)$ etc B_{is}^0 are values of δ_i and B_{is} are values of $f(x_i)$ which is the source function of the problem being considered.

Numerical Examples

This section describes the demonstration of the suggested method mentioned above on some examples. The problems are solved using the Bernstein polynomials as basic functions. The examples are solved to illustrate the computational cost accuracy and efficiency of the suggested method using Maple 18.

Example 1: Consider the following fractional Integro-differential

$$D^{\frac{3}{4}}u(x) = -\frac{x^2 e^x}{5}u(x) + \frac{6x^{2.25}}{\Gamma(3.25)} + e^x \int_0^x tu(t)dt \quad (21)$$

Subject to $u(0) = 0$. The exact solution is $u(x) = x^3$

Applying the Bernstein collocation method with the aid of Bernstein polynomials to fractional

integro-differential equation (21). The numerical result is showing in figure 1. The solution obtained using the suggested method is in excellent agreement with the already exact solution and show that this approach can solve the problem effectively. Comparisons are made

between approximate solution and exact solution to illustrate the validity and the great potential of the proposed technique. Also, from our numerical result, we can see that the solution is more accuracy than the obtained result in (Rawashdeh, 2006).

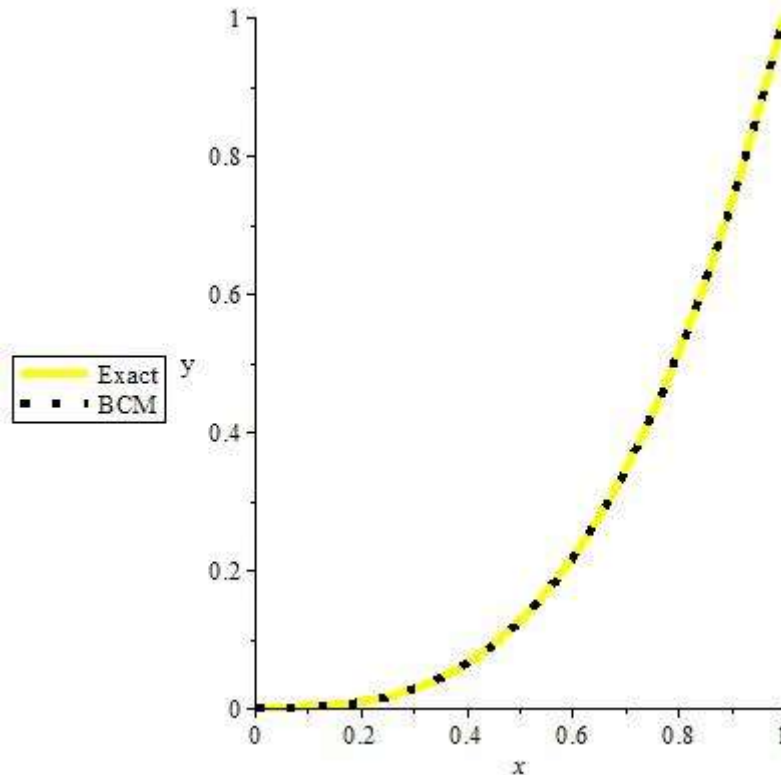


Fig. 1: Comparison between the approximate solution and the exact solution.

Example 2: Consider the following fractional Integro-differential

$$D_{\frac{1}{2}}u(x) = u(x) + \frac{8x^{2.25}}{3\Gamma(0.5)} - x^2 - \frac{1}{2}x^3 + \int_0^x tu(t)dt \quad (22)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^2$

Similarly as in example 1, applying the Bernstein collocation method with the aid of Bernstein polynomials to fractional integro-differential equation (22). The numerical result is showing in figure 2 and

the obtained approximate solution is the same as the exact solution. The solution obtained using the suggested method is in excellent agreement with the already exact solution. Comparisons are made between the approximate solution and the exact solution to illustrate the validity and the great potential of the proposed technique. Also, from our numerical result, we can see that the solution is more accurate than the obtained result in (Mohamed et. al., 2016).

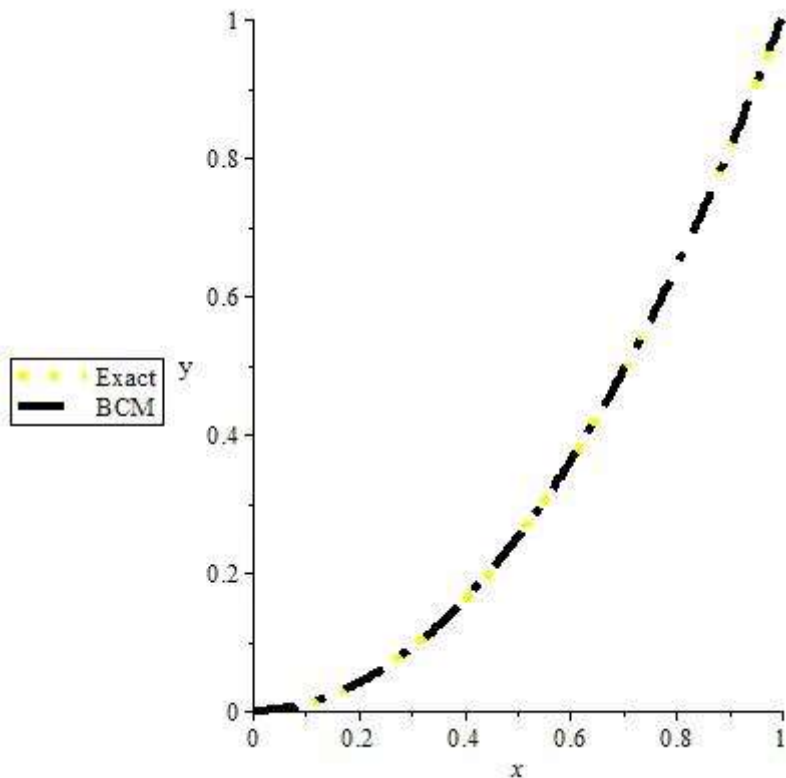


Fig. 2: Comparison between the approximate solution and the exact solution

Example 3: Consider the following fractional Integro-differential

$$D^{\frac{1}{2}}u(x) = (\cos(x) - \sin(x))u(x) + f(x) + \int_0^x x \sin(t)u(t)dt \quad (23)$$

$$f(x) = \frac{2x^{1.5}}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)}x^{0.5} + x(\cos(x) - x \sin(x) + x^2 \cos(x)) \quad (24)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^2 + x$

Similarly, as in example 1 and 2 applying the collocation method with the aid of Bernstein polynomials at to fractional integro-differential equation (23). The numerical result is showing in

figure 3 and we obtain the approximate solution which is the same the exact solution. The solution obtained using the suggested method is in excellent agreement with the already exact solution. Comparisons are made between the approximate solution and the exact solution to illustrate the validity and the great potential of the proposed technique. Also, from our numerical result, we can see that the solution is more accurate than the one obtained in (Mohamed et. al., 2016)

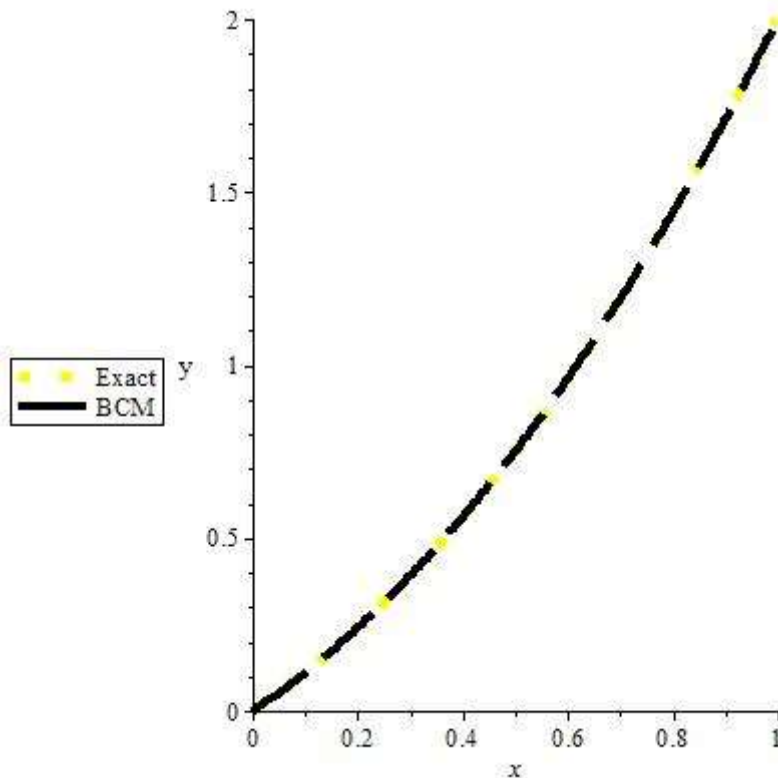


Fig. 3: Comparison between the approximate solution and the exact solution.

CONCLUSION

In this study, we introduced an accurate numerical method for solving fractional integro-differential equations. Bernstein polynomials are used as basic functions to find the solution of FIDEs. Some examples were solved using the suggested method. The results obtained compared with (Rawashdeh, 2006) and (Mohamed et. al., 2016) showed that the

suggested method is more accurate than and (Mohamed et. al., 2016)

Hence, calculation showed that the suggested method is an efficient technique in finding a very good solution for this type of equation. Also, the results were presented in graphical forms to further demonstration of the method.

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