



## Numerical Solution of Fourth order Integro-Differential Equations by Least Square Approximation Method using Chebyshev Polynomials as Basis Function

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### ABSTRACT

*In this paper, Least square approximation method is used as numerical solution of fourth order integro-differential equations using Chebyshev polynomials of the first kind as basis function. The method assumed an approximate solution using Chebyshev polynomial functions which are then substituted into the problem considered. Then the like terms of the unknown coefficients are collected and simplified. The resulting equation is minimized using the least square approximation, thus resulted into linear algebraic systems of equations which are then solved by Gaussian Elimination method, to obtain the unknown constants substituted back into the assumed approximate solution to get the required approximate solution. Numerical solutions are given to illustrate the accuracy of the methods discussed in the work. Also, absolute errors of the problem are presented in tabular forms..*

### ARTICLE INFO

Article History

Received: September, 2022

Received in revised form: December, 2022

Accepted: January, 2023

Published online: March, 2023

**KEYWORDS:** Gaussian Elimination method, Least square method, Integro - Differential equation

### INTRODUCTION

The concept of integral and integro-differential equation have motivated large amount of research work in recent years any numerical methods have been applied to solve these equation such as Variational iteration, Homotopy perturbation, Adomain decomposition, and so on. Integro-differential equations appear in many scientific application. The integro-differential equation contains both integral and differential operators. The derivatives of the unknown functions may appear to any order. Mathematical modelling of real-life problems usually results in functional equations, e.g.

partial differential equations, integral and integro-differential equations, stochastic equations and others. Many Mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in fluid dynamics, biological models and chemical kinetics; Kythe and Puri (2002), obtained combinatorial method for linear integral equations, Hashim (2006) derived Adomian decomposition method for solving Boundary value problems for fourth order integro-differential equations. Laeli and Maalek Ghaini (2012) delved into Numerical solution of Volterra-Fredholm integral equations by moving least square

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method and Chebyshev polynomials. Also, Sweilam et.al. (2013) obtained Numerical and Analytical Study for Fourth-Order Integro-Differential equations using a Pseudospectral Method. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution.

In this work, a method based on Least Square Approximation method and Chebyshev polynomial is presented for addressing the following fourth order integro-differential equation:

$$u^{(n)}(x) = f(x) + \gamma \int_a^b k(x, t)u(t)dt, \quad 0 \leq x, t \leq 1.$$

Together with the associated conditions:

$$y^i(a) = \alpha_i, i = 0, 1, 2, 3$$

Where  $\gamma, \alpha_i, i = 0, 1, 2, 3$ , are real constants, and  $f(x)$  and  $k(x, t)$  are given and can be approximated by Taylor polynomials, which plays significant role in many field of applied science. Therefore, the problem has attracted much attention and has been studied by many authors.

The method is proposed by using "shifted Chebyshev" for solving the integro-differential equations: Abbasbandy and Shivanian (2009) extended the variational iteration method for solving Fredholms nth-order integro-differential equations. Wazwaz (2001) main objective was only to obtain

$$u^{(iv)}(x) = f(x) + \lambda \int_a^b k(x, t)u(t)dt \quad \dots\dots\dots(1)$$

Let the assumed equation be

$$u_n(x) = \sum_{i=0}^n a_i T_i(x), \quad 0 \leq x \leq 1 \\ = a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + a_3 T_3(x) + \dots + a_n T_n(x) \dots\dots\dots(2)$$

Differentiating (2) with respect to  $x$  four-times, we have

$$u_n^{(iv)}(x) = a_0 T_0^{(iv)}(x) + a_1 T_1^{(iv)}(x) + a_3 T_3^{(iv)}(x) + \dots + a_n T_n^{(iv)}(x) \dots\dots\dots(3)$$

Replacing "x" by "t" in (2), we get

$$u_n(t) = a_0 T_0(t) + a_1 T_1(t) + a_2 T_2(t) + a_3 T_3(t) + \dots + a_n T_n(t) \dots\dots\dots(4)$$

the exact solutions to two fourth order integro-differential equations. Several methods to solve Integro-differential equation have been given such as "the Adomain decomposition method", "Variational iteration method", "moving least square method" and so on. Other numerical methods be found in (Peter, 2017, 2018, Peter et.al. 2019, 2020, and 2021, Oyedepo et. al., 2021).

In this work, we present a Least Square Approximation method for solving (1.1) and obtain an accurate approximate numerical solution.

The arrangement of this paper goes thus: In section 2 we review some basic definitions that will guide the readers in the understanding of the paper, in section 3 we demonstrated the numerical scheme on the general class of the problem, in section 4 we solve three numerical examples while in the 4th section we discussed the results and the give the summary of the whole work.

## METHODOLOGY

### Description of the Least Square Approximation method on the general problem considered

Consider the general fourth order integro-differential equation of the form.

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Substituting (3) and (4) into (1), we obtain

$$\sum_{i=0}^n a_0 T_i^{(iv)}(x) = f(x) + \lambda \int_a^b k(x, t) (\sum_{i=0}^n a_i T_i(t)) dt \dots \dots (5)$$

$$a_0 T_0^{(iv)} + a_1 T_1^{(iv)} + \dots + a_n T_n^{(iv)}(x) = f(x) + \lambda \int_a^b (a_0 T_0(t) + a_1 T_1(t) + \dots + a_n T_n(t)) dt \dots \dots (6)$$

The residual equation from (5) is written as

$$R(a_0, a_1, a_2, \dots, a_n) = \{a_0 T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x) - \lambda \int_a^b k(x, t) (a_0 T_0(t) + a_1 T_1(t) + \dots + a_n T_n(t)) dt - f(x)\} \dots \dots (7)$$

Applying Least Square Approximation method, Let

$$I(a_0, a_1, a_2, \dots, a_n) = \int_0^1 [R(a_0, a_1, a_2, \dots, a_n)]^2 w(x) dx$$

Where  $R(a_0, a_1, a_2, \dots, a_n)$  is equivalent to (6) and is called the residual equation,  $w(x)$  is the weight function and is taken to be 1 throughout this project work for simplicity.

$$I(a_0, a_1, a_2, \dots, a_n) = \int_0^1 \{a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \dots + a_n T_n(x) - \lambda \int_a^b k(x, t) (a_0 T_0(t) + a_1 T_1(t) + a_2 T_2(t) + \dots + a_n T_n(t)) dt - f(x)\}^2 \dots \dots (8)$$

So, finding the values of  $a_i, i = 0, 1, 2, \dots, n$ , which minimizes  $I$  is equivalent to finding the best approximation for the solution of the integro-differential equation (1).

The minimum value of  $I$  is obtained by setting

$$\frac{\partial I}{\partial a_i} = 0, \quad i = 0, 1, \dots, n \dots \dots (9)$$

Applying (9) to (8), we obtain

$$\begin{aligned} \frac{\partial I}{\partial a_0} = & -2 \int_0^1 \{T_0^{(iv)}(x) - \lambda \int_a^b k(x, t) T_0(t) dt\} a_0 + \{T_1^{(iv)}(x) - \lambda \int_a^b k(x, t) T_1(t) dt\} a_1 \\ & + \{T_2^{(iv)}(x) - \lambda \int_a^b k(x, t) T_2(t) dt\} a_2 + \dots + \{T_n^{(iv)}(x) - \lambda \int_a^b k(x, t) T_n(t) dt\} a_n \\ & - f(x) \{T_0^{(iv)}(x) - \lambda \int_a^b k(x, t) T_0(t) dt\} dx = 0 \dots \dots (10) \end{aligned}$$

$$\begin{aligned} \frac{\partial I}{\partial a_1} = & -2 \int_0^1 \{T_0^{(iv)}(x) - \lambda \int_a^b k(x, t) T_0(t) dt\} a_0 + \{T_1^{(iv)}(x) - \lambda \int_a^b k(x, t) T_1(t) dt\} a_1 \\ & + \{T_2^{(iv)}(x) - \lambda \int_a^b k(x, t) T_2(t) dt\} a_2 + \dots + \{T_n^{(iv)}(x) - \lambda \int_a^b k(x, t) T_n(t) dt\} a_n \\ & - f(x) \{T_1^{(iv)}(x) - \lambda \int_a^b k(x, t) T_1(t) dt\} dx = 0 \dots \dots (11) \end{aligned}$$

$$\begin{aligned} \frac{\partial I}{\partial a_2} = & -2 \int_0^1 \{T_0^{(iv)}(x) - \lambda \int_a^b k(x, t) T_0(t) dt\} a_0 + \{T_1^{(iv)}(x) - \lambda \int_a^b k(x, t) T_1(t) dt\} a_1 \\ & + \{T_2^{(iv)}(x) - \lambda \int_a^b k(x, t) T_2(t) dt\} a_2 + \dots + \{T_n^{(iv)}(x) - \lambda \int_a^b k(x, t) T_n(t) dt\} a_n \\ & - f(x) \{T_2^{(iv)}(x) - \lambda \int_a^b k(x, t) T_2(t) dt\} dx = 0 \dots \dots \end{aligned}$$

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$$-f(x)\{T_2^{(iv)}(x) - \lambda \int_a^b k(x,t)T_2(t)dt\}dx = 0 \dots \dots \dots (12)$$

$$\begin{aligned} \frac{\partial I}{\partial a_n} = & -2 \int_0^1 \{T_0^{(iv)}(x) - \lambda \int_a^b k(x,t)T_0(t)dt\}a_0 + \{T_1^{(iv)}(x) - \\ & \lambda \int_a^b k(x,t)T_1(t)dt\}a_1 \\ & + \{T_2^{(iv)}(x) - \lambda \int_a^b k(x,t)T_2(t)dt\}a_2 + \dots + \{T_n^{(iv)}(x) - \\ & \lambda \int_a^b k(x,t)T_n(t)dt\}a_n \\ & -f(x)\{T_n^{(iv)}(x) - \lambda \int_a^b k(x,t)T_n(t)dt\}dx = 0 \dots \dots \dots (13) \end{aligned}$$

Evaluating the integrals in (10)-(13) gives (n+1) algebraic linear equations in (n+1) unknown constants  $a_i (i \geq 0)$ , which is then solved using Gaussian Elimination method to obtain the unknown constants  $a_0, a_1, \dots, a_n$  which are then substituted back into (2) to obtain the required approximate solution.

## Chebyshev Polynomials in the Interval [A,B]

### Chebyshev polynomial of the first kind

Chebyshev polynomial of the first kind  $T_n(x)$  is defined as

$$T_n(x) = \cos(ncos^{-1}x); \quad [-1,1]$$

or

$$T_n(x) = \cos n\theta \dots \dots \dots (14)$$

where,  $\theta = cos^{-1}x$  and  $x = \cos\theta$

From (14), we obtain

$$\begin{aligned} T_0(x) &= \cos(0\theta) = 1 \\ T_1(x) &= \cos(1\theta) = x \\ T_2(x) &= \cos(2\theta) = 2\cos^2\theta - 1 = 2x^2 - 1 \end{aligned}$$

From trigonometry, we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \dots \dots \dots (15)$$

let  $n \geq 1$ , from (15), we obtain

$$T_{n+1}(x) = \cos(\theta(n+1)) = \cos(n\theta + \theta) = \cos\theta\cos n\theta - \sin\theta\sin n\theta \dots \dots (16)$$

$$T_{n-1}(x) = \cos(\theta(n-1)) = \cos(n\theta - \theta) = \cos\theta\cos n\theta + \sin\theta\sin n\theta \dots \dots (17)$$

Adding (16) and (17) and after simplifying, we obtain

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \dots \dots \dots (18)$$

(18) is the recurrence relation which can be used to successively compute all  $T_n(x)$ .

Since we know  $T_0(x)$  and  $T_1(x)$ . The first seven Chebyshev Polynomials  $T_n(x)$  on interval  $[-1,1]$  are:

$$\left. \begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \end{aligned} \right\} \dots \dots \dots (19)$$

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To use Chebyshev polynomials on interval  $[0,1]$ , we need to change the domain using the following substitution.

$$X = 2x - 1, \quad 0 \leq x \leq 1$$

So, the shifted Chebyshev polynomials  $T_n^*(x)$  on  $[0,1]$  are obtained as follows:

$$\left. \begin{aligned} T_n^*(x) &= T_n(2x - 1) \\ T_0^*(x) &= 1 \\ T_1^*(x) &= 2x - 1 \\ T_2^*(x) &= 8x^2 - 8x + 1 \\ T_3^*(x) &= 32x^3 - 48x^2 + 18x - 1 \\ T_4^*(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ T_5^*(x) &= 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1 \\ T_6^*(x) &= 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1 \end{aligned} \right\} \dots\dots\dots(20)$$

### DEMONSTRATION OF LINEAR CASES LEAST SQUARE APPROXIMATION METHOD ON SOME NUMERICAL EXAMPLES.

In this section, we demonstrated the proposed method discussed above on some examples.

**Numerical Example Example 1** Consider the linear fourth order Integro-differential equation as in (1.1) with

$$f(x) = \frac{5!}{\Gamma(2)}x + x^5 \left(1 + \frac{1}{7}x^2\right), \gamma = -1, g(x) = -x \text{ and } h(x) = 0,$$

i. e

$$u^{iv}(x) = \frac{5!}{\Gamma(2)}x + x^5 \left(1 + \frac{1}{7}x^2\right) - u(x) + \int_0^x tu(t)dt, \dots\dots\dots(21)$$

subject to

$$u^{(i)} = 0, \quad i = 0, 1, 2, 3$$

with exact solution

$$u(x) = x^5.$$

### Solution

We have solved the above problem for various values of  $n$  in order to compare the results obtained with the exact solution. Also, Tabulated Error representations of the result are presented, for various values of  $n(n = 6)$ . Thus the approximate solution given in equation (2) becomes

$$\begin{aligned} u_6(x) &= a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3 \\ &+ a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 1280x^4 + 1120x^3 \\ &- 400x^2 + 50x - 1) + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 \\ &- 72x + 1) \dots\dots\dots(22) \end{aligned}$$

differentiating (22) with respect to four times, we have

$$3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + 165888) \dots\dots\dots(23)$$

replacing  $x$  by  $t$  in (22) we have

$$\begin{aligned} &a_0 + a_1(2t - 1) + a_2(8t^2 - 8t + 1) + a_3(32t^3 - 48t^2 + 18t - 1) + \\ &a_4(128t^4 \\ &- 256t^3 + 160t^2 - 32t + 1) + a_5(512t^5 - 1280t^4 + 1120t^3 - 400t^2 + \end{aligned}$$

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$50t - 1)$

$$+a_6(2048t^6 - 6144t^5 + 6912t^4 - 3584t^3 + 840t^2 - 72t + 1).....(24)$$

putting (22), (23) and (24) in (21), we obtain the residual equation

$$\begin{aligned} R(a_0, a_1, \dots, a_6) = & 3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x \\ & + 165888) - 120x - x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) \\ & + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + 18x - 1) + a_4(128x^4 \\ & - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 1280x^4 + 1120x^3 \\ & - 400x^2 + 50x - 1) + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 \\ & + 840x^2 - 72x + 1) + 256a_6x^8 + \frac{1}{7}(512a_5 - 6144a_6)x^7 \\ & + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - 256a_4 \\ & + 1120a_5 - 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 \\ & + 840a_6)x^4 + \frac{1}{3}(2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6)x^3 \\ & + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)x^2 = 0 \end{aligned}$$

Then,

$$\begin{aligned} I(a_0, a_1, \dots, a_6) = & \int_0^1 [3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + 165888) \\ & - 120x - x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) \\ & + a_3(32x^3 - 48x^2 + 18x - 1) + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) \\ & + a_5(512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1) \\ & + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) + 256a_6x^8 \\ & + \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 \\ & + \frac{1}{5}(32a_3 - 256a_4 + 1120a_5 - 3584a_6)x^5 \\ & + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 + \frac{1}{3}(2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6)x^3 \\ & + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)x^2]^2 dx = 0 \end{aligned} \quad (25)$$

In order to minimize equation (25), applying equation (9) we have the following equations.

$$\begin{aligned} \frac{\partial I}{\partial a_0} = & 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\ & 165888) - 120x \\ & - x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\ & 18x - 1) \\ & + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 + \\ & 1120x^3 - 400x^2 + 50x - 1) \\ & + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) + \\ & 256a_6x^8 \\ & + \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\ & 256a_4 \\ & + 1120a_5 - 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 \end{aligned}$$

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$$\begin{aligned}
 & + \frac{1}{3}(2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + \\
 & a_4 - a_5 + a_6)x^2) \\
 & (1 + \frac{1}{2}x^2) = 0 \dots \dots \dots (26) \\
 & \frac{\partial I}{\partial a_1} = 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\
 & 165888) - 120x \\
 & -x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\
 & 18x - 1) \\
 & + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 + \\
 & 1120x^3 - 400x^2 + 50x - 1) \\
 & + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) + \\
 & 256a_6x^8 \\
 & + \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\
 & 256a_4 \\
 & + 1120a_5 - 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 \\
 & + \frac{1}{3}(2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + \\
 & a_4 - a_5 + a_6)x^2) \\
 & (2x - 1 + \frac{2}{3}x^3 - \frac{1}{2}x^2) = 0 \dots \dots \dots (27) \\
 & \frac{\partial I}{\partial a_2} = 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\
 & 165888) - 120x \\
 & -x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\
 & 18x - 1) \\
 & + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 + \\
 & 1120x^3 - 400x^2 + 50x - 1) \\
 & + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) + \\
 & 256a_6x^8 \\
 & + \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\
 & 256a_4 \\
 & + 1120a_5 - 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 \\
 & + \frac{1}{3}(2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + \\
 & a_4 - a_5 + a_6)x^2) \\
 & (\frac{7}{2}x^2 - 8x + 1 + 2x^4 - \frac{8}{3}x^3) = 0 \dots \dots \dots (28) \\
 & \frac{\partial I}{\partial a_3} = 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\
 & 165888) - 120x \\
 & -x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\
 & 18x - 1) \\
 & + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 +
 \end{aligned}$$





$$\begin{aligned}
 &1120x^3 - 400x^2 \\
 &1) + 256a_6x^8 \\
 &+ \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\
 &256a_4 + 1120a_5 \\
 &- 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 + \frac{1}{3}(2a_1 - \\
 &8a_2 + 18a_3 - 32a_4 \\
 &+ 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)x^2)(38x^3 - \\
 &\frac{97}{2}x^2 + 18x - 1 \\
 &+ \frac{32}{5}x^5 - 12x^4) = 0 \dots \dots \dots (29)
 \end{aligned}$$

$$\begin{aligned}
 &\frac{\partial I}{\partial a_4} = 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\
 &165888) - 120x \\
 &- x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\
 &18x - 1) \\
 &+ a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 + \\
 &1120x^3 - 400x^2 \\
 &1) + 256a_6x^8 \\
 &+ \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\
 &256a_4 + 1120a_5 \\
 &- 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 + \frac{1}{3}(2a_1 - \\
 &8a_2 + 18a_3 - 32a_4 \\
 &+ 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)x^2)(3073 + \\
 &168x^4 - \frac{800}{3}x^3 \\
 &+ \frac{321}{2}x^2 - 32x + \frac{64}{3}x^6 - \frac{256}{5}x^5) = 0 \dots \dots \dots (30)
 \end{aligned}$$

$$\begin{aligned}
 &\frac{\partial I}{\partial a_5} = 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\
 &165888) - 120x \\
 &- x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\
 &18x - 1) \\
 &+ a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 + \\
 &1120x^3 - 400x^2 \\
 &1) + 256a_6x^8 \\
 &+ \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\
 &256a_4 + 1120a_5 \\
 &- 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 + \frac{1}{3}(2a_1 -
 \end{aligned}$$

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$$\begin{aligned}
 &8a_2 + 18a_3 - 32a_4 \\
 &\quad + 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)x^2)(61490x - \\
 &30721 + 736x^5 \\
 &\quad - 1380x^4 + \frac{3410}{3}x^3 - \frac{801}{2}x^2 + \frac{512}{7}x^7 - \frac{640}{3}x^6) = 0 \dots \dots \dots (31) \\
 &\frac{\partial I}{\partial a_6} = 2(3072a_4 + a_5(61440x - 30720) + a_6(737280x^2 - 737280x + \\
 &165888) - 120x \\
 &\quad - x^5 \left(1 + \frac{1}{7}x^2\right) + a_0 + a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + \\
 &18x - 1) \\
 &\quad + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 - 128x^4 + \\
 &1120x^3 - 400x^2 \\
 &\quad + 50x - 1) + a_6(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + \\
 &1) + 256a_6x^8 \\
 &\quad + \frac{1}{7}(512a_5 - 6144a_6)x^7 + \frac{1}{6}(128a_4 - 1280a_5 + 6912a_6)x^6 + \frac{1}{5}(32a_3 - \\
 &256a_4 + 1120a_5 \\
 &\quad - 3584a_6)x^5 + \frac{1}{4}(8a_2 - 48a_3 + 160a_4 - 400a_5 + 840a_6)x^4 + \frac{1}{3}(2a_1 - \\
 &8a_2 + 18a_3 - 32a_4 \\
 &\quad + 50a_5 - 72a_6)x^3 + \frac{1}{2}(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)x^2) \\
 &\quad \left(\frac{1476241}{2}x^2 - 737352x + 165889 + 3200x^6 - \frac{34304}{5}x^5 + 7122x^4 - \right. \\
 &3608x^3 + 256x^8 - \left. \frac{6144}{7}x^7\right) = 0 \dots \dots \dots (32)
 \end{aligned}$$

Integrating equation (26) - (32) with respect to  $x$  over the interval  $[0,1]$ , we have the following equations

$$-\frac{18061}{120} + \frac{144764833}{1386}a_6 + \frac{967676}{189}a_5 + \frac{2709449}{378}a_4 - \frac{4}{35}a_3 - \frac{125}{126}a_2 + \frac{8}{45}a_1 + \frac{83}{30}a_0 = 0 \dots \dots \dots (33)$$

$$-\frac{3517007}{83160} + \frac{42577877}{10395}a_6 + \frac{89413217}{4158}a_5 - \frac{4}{315}a_4 - \frac{857}{1890}a_3 - \frac{19}{315}a_2 + \frac{31}{42}a_1 + \frac{8}{45}a_0 = 0 \dots \dots \dots (34)$$

$$\begin{aligned}
 &\frac{964939}{16632} + \frac{37401556303}{1351350}a_6 - \frac{10644461}{3465}a_5 - \frac{276795221}{103950}a_4 + \frac{191}{4725}a_3 + \frac{229}{2}a_2 - \frac{19}{315}a_1 \\
 &\quad - \frac{125}{126}a_0 = 0 \dots \dots \dots (35)
 \end{aligned}$$

$$\begin{aligned}
 &\frac{9616879}{360360} - \frac{395366287}{75075}a_6 - \frac{3677003219}{270270}a_5 + \frac{416}{51975}a_4 + \frac{1415}{1386}a_3 + \frac{191}{4725}a_2 \\
 &\quad - \frac{857}{1890}a_1 - \frac{4}{35}a_0 = 0 \dots \dots \dots (36)
 \end{aligned}$$

$$\begin{aligned}
 &-\frac{2798242870411}{7567560} + \frac{2499157229440457}{9459450}a_6 + \frac{138379048}{315315}a_5 + \frac{3643572677977}{193050}a_4 + \frac{416}{51975}a_3 \\
 &\quad - \frac{276795221}{103950}a_2 - \frac{4}{315}a_1 + \frac{2709449}{378}a_0 = 0, \dots \dots \dots (37)
 \end{aligned}$$

$$\frac{720060498137}{582120} + \frac{88058993}{85995}a_6 + \frac{170037731905669}{270270}a_5 + \frac{138379048}{315315}a_4 - \frac{3677003219}{270270}a_3$$

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$$-\frac{10644461}{3465}a_2 + \frac{89413217}{4158}a_1 + \frac{967676}{189}a_0 \dots \dots \dots (38)$$

$$-\frac{13100721204049}{2522520} + \frac{522053100908937107}{53603550}a_6 + \frac{88058993}{85995}a_5 + \frac{2499157229440457}{9459450}a_4 \\ - \frac{395366287}{75075}a_3 + \frac{37401556303}{1351350}a_2 + \frac{42577877}{10395}a_1 + \frac{144764833}{1386}a_0 \dots \dots \dots (39)$$

The various values obtained in solving equation (33)-(39) are as follow:

$$a_0 = .2460937500, a_1 = .4101562500, a_2 = .2343750000, a_3 = 0.08789062500, \\ a_4 = 0.01953125000, a_5 = 0.001953125000, a_6 = 0.$$

Those values are substituted back into equation (21) and after simplifying we have the approximate solution as

$$u_6(x) = x^5$$

**Numerical Example Example 2** Consider the linear fourth order integro-differential equation

$$u^{iv}(x) = 3x + 2u''(x) + \int_0^1 xu(t)dt, \dots \dots (40)$$

subject to

$$u(0) = u(1) = 0, \quad u''(0) = 0, \quad u''(1) = -\frac{84}{47}$$

with exact solution

$$u(x) = \frac{61}{47}x - \frac{14}{47}x^3.$$

**Solution**

The values for various  $a_i$ ,  $i = 0, \dots, 6$  are substituted back into equation (41) and after simplifying, we have the approximate solution to be

$$u_6(x) = \frac{61}{47}x - \frac{14}{47}x^3$$

**Numerical Example Example 3** Consider the linear fourth order integro-differential equation

$$u^{iv}(x) = 1 - x + \sin x + \int_0^{\frac{\pi}{2}} (x - t)u(t)dt, \dots \dots (41)$$

subject to

$$u(0) = u'' = 0, \quad u'(0) = -u'''(0) = 1$$

with exact solution

$$u(x) = \sin x.$$

**Solution**

The values for various  $a_i$ ,  $i = 0, \dots, 6$  are substituted back into equation (4.120) and after simplifying, we have the approximate solution to be

$$u_6(x) = -5.63280010^{-1} + 1.000000001x - 1.10^{-1}x^2 - \\ 0.1666666666x^3 \\ - 0.00031118328x^4 + 0.009094160015x^5 - 0.0006540605440x^6$$



## Table of results

In this section, the results obtained in the Numerical examples are represented in tabular form.

**Table 1:** Table of Error of Example 1

$x$	Exact	Approximate	error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	0.0000100000	0.0000100000	0.0000000000
0.2	0.0003200000	0.0003200000	0.0000000000
0.3	0.0024300000	0.0024300000	0.0000000000
0.4	0.0102400000	0.0102400000	0.0000000000
0.5	0.0312500000	0.0312500000	0.0000000000
0.6	0.0777600000	0.0777600000	0.0000000000
0.7	0.1680700000	0.1680700000	0.0000000000
0.8	0.3276800000	0.3276800000	0.0000000000
0.9	0.5904900000	0.5904900000	0.0000000000
1.0	1.0000000000	1.0000000000	0.0000000000

**Table 2:** Table of Error of Example 2

$x$	Exact	Approximate	error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	0.1294893617	0.1294893617	0.0000000000
0.2	0.2571914894	0.2571914894	0.0000000000
0.3	0.3813191489	0.3813191489	0.0000000000
0.4	0.5000851064	0.5000851064	0.0000000000
0.5	0.6117021276	0.6117021276	0.0000000000
0.6	0.7143829787	0.7143829787	0.0000000000
0.7	0.8063404255	0.8063404255	0.0000000000
0.8	0.8857872337	0.8857872337	0.0000000000
0.9	0.9509361698	0.9509361698	0.0000000000
1.0	0.9999999996	0.9999999996	0.0000000000

**Table 3:** Table of Error of Example 3

$x$	Exact	Approximate	error
0.0	0.0000000000	-0.0000000006	$5.6328 \times 10^{-10}$
0.1	0.0998334166	0.0998333920	$2.4608 \times 10^{-08}$
0.2	0.1986693308	0.1986690366	$2.9416 \times 10^{-07}$
0.3	0.2955202067	0.2955191011	$1.1056 \times 10^{-06}$
0.4	0.3894183423	0.3894158120	$2.5303 \times 10^{-06}$
0.5	0.4794255386	0.4794211904	$4.3482 \times 10^{-06}$
0.6	0.5646424734	0.5646363167	$6.1567 \times 10^{-06}$
0.7	0.6442176872	0.6442101243	$7.5630 \times 10^{-06}$

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0.8	0.7173560909	0.7173477225	$8.3684 \times 10^{-06}$
0.9	0.7833269096	0.7833182487	$8.6608 \times 10^{-06}$
1.0	0.8414709848	0.8414622495	$8.7349 \times 10^{-06}$

## CONCLUSION

In this work, the Least Squares Approximation Method is used to solve some numerical examples on fourth order integro-differential equations using the shifted Chebyshev polynomials as basis function. From the table of results presented, we observe

that as the  $n$  increases the approximate solution is very close/same as the exact solution. This shows that the Least square approximation method is suitable for solving the classes of fourth order integro-differential problems discussed in this work.

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